FINITE DIMENSIONAL ALGEBRAS

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Corollary. *This document is provided as is, with potentially numerous typos and errors, without warranty of any kind.*
Chapter 1

Algebras and modules

1.1 Algebras

For all this course, the letter \( K \) will denote a (commutative) field.

**Definition 1.1.1 (\( K \)-algebra).** A \( K \)-algebra is a quadruple \( (A, +, \cdot, \ast) \) where

1. \( (A, +, \cdot) \) is a ring with unit \( 1_A \),
2. \( (A, +, \ast) \) is a \( K \)-vector space,
3. \( \forall \lambda \in K, \forall a, b \in A \) we have \( (\lambda \ast a) \cdot b = \lambda \ast (a \cdot b) = a \cdot (\lambda \ast b) \).

In order to simplify notations, we’ll denote \( \lambda a = \lambda \ast a \).

**Remark 1.1.2.** Since a \( K \)-algebra \( A \) is also a \( K \)-vector space, we call the *dimension* of \( A \) its dimension as a \( K \)-vector space.

**Example 1.1.3.** \( \mathbb{C} \) is a \( \mathbb{C} \)-algebra of dimension 1, and a \( \mathbb{R} \)-algebra of dimension 2.

**Definition 1.1.4 (\( K \)-algebra, alternate definition).** A \( K \)-algebra is a quadruple \( (A, +, \cdot, \phi) \) where :

1. \( (A, +, \cdot) \) is a ring with unit \( 1_A \),
2. \( \phi : K \to Z(A) \) is a ring homomorphism.

**Proposition 1.1.5.** *The two definitions of \( K \)-algebra are equivalent.*

**Proof.** Take \( A \) a \( K \)-algebra in the sense of definition 1.1.1, and define

\[
\phi : K \to Z(A) \\
\lambda \mapsto \lambda \ast 1_A.
\]
CHAPTER 1. ALGEBRAS AND MODULES

Condition 3 of definition 1.1.1 states that the above morphism is well defined. Conversely, take $B$ a $\mathbb{K}$-algebra in the sense of definition 1.1.4, and define the $*$ operation by

$$\star : \mathbb{K} \times B \longrightarrow B$$

$$(\lambda, b) \mapsto \lambda \star b = \phi(\lambda) \cdot b.$$ 

Since $\phi$ takes its image in $Z(B)$, the above operation satisfies condition 2 and 3 of definition 1.1.1.

Remark 1.1.6. The morphism $\phi : \mathbb{K} \longrightarrow Z(A)$ is always injective, unless $A = 0$.

Examples 1.1.7. 1. $\mathbb{K}$ is a $\mathbb{K}$-algebra of dimension 1;

2. $M_n(\mathbb{K})$ with the morphism

$$\phi : \mathbb{K} \longrightarrow M_n(\mathbb{K})$$

$$\lambda \longmapsto \lambda I_n$$

is a $\mathbb{K}$-algebra of dimension $n^2$;

3. If $(A, +, \cdot, *)$ is a $\mathbb{K}$ algebra, the opposite algebra $(A^{\text{op}}, +, \odot, *)$ is defined as follows:

(a) $(A, +, *) = (A^{\text{op}}, +, *)$ as $\mathbb{K}$-vector spaces,

(b) $\forall a, b \in A^{\text{op}}, a \odot b = b \cdot a$;

4. If $G$ is a group, the group algebra $\mathbb{K}G$ is a $\mathbb{K}$-algebra of dimension $|G|$;

5. The previous example also works if $G$ is a monoid, and the result is called a monoid algebra;

6. A quiver $Q = (V, E, \varepsilon)$ is a finite directed (multi-)graph:

(a) $V$ is a finite set of vertices,

(b) $E$ is a finite set of directed edges,

(c)

$$\varepsilon : E \longrightarrow V \times V$$

$$e \mapsto (\varepsilon_0(e), \varepsilon_1(e)),$$

where $\varepsilon_0(e)$ and $\varepsilon_1(e)$ are the origin and target of $e$ respectively.
For every vertex \( v \in V \), we define a path \( l_v \) of length 0 from \( v \) to \( v \), which is called a lazy path. Let \( \mathbb{K}Q \) be the \( \mathbb{K} \)-vector space with basis the set of paths of \( Q \). The multiplication is induced by the concatenation of paths:

\[
(e_1, \ldots, e_n) \cdot (f_1, \ldots, f_m) = \begin{cases} (e_1, \ldots, e_n, f_1, \ldots, f_m) & \text{if } e_1(e_n) = e_0(f_1) \\ 0 & \text{otherwise,} \end{cases}
\]

and the neutral element is given by \( \sum_{v \in V} l_v \).

**Definition 1.1.8** (Morphism of algebras). Let \( A \) and \( B \) be two \( \mathbb{K} \)-algebras. A morphism of algebras \( f : A \rightarrow B \) is a \( \mathbb{K} \)-linear ring morphism. An algebra isomorphism is a bijective algebra morphism. Starting with definition 1.1.4, \( f \) is an algebra morphism if it is a morphism of ring such that the following diagram commutes:

\[
\begin{array}{ccc}
\phi_A & \mathbb{K} & \phi_B \\
\downarrow & \phi & \downarrow \\
A & f & B.
\end{array}
\]

### 1.2 Modules

#### 1.2.1 Basic definitions

**Definition 1.2.1** (\( R \)-Module). Let \( R \) be a ring (with unit). A left \( R \)-module is a triple \((M, +, \cdot)\) where

1. \((M, +)\) is an abelian group,
2. \( \cdot : R \times M \rightarrow M \) is such that \( \forall r, s \in R, \forall m, m' \in M \)

\[
(r + s) \cdot m = r \cdot m + s \cdot m \\
1_r \cdot m = m.
\]

Again, in order to simplify notation, we’ll denote \( rm = r \cdot m \). A right \( R \)-module is defined in a similar way, but with \( \cdot : M \times R \rightarrow M \). If \( M \) is a left \( R \)-module, we emphasize this structure with the notation \( _RM \). Similarly, if \( M \) is a right \( R \)-module, we note \( M_R \).

**Properties 1.2.2.**

\[
\begin{align*}
r0_M &= 0_M \\
0_{RM} &= 0_M \\
r(-m) &= -(rm) = (-r)m.
\end{align*}
\]
Remark 1.2.3. Since algebras are rings, we have a definition of module \( M \) over an \( \mathbb{K} \)-algebra \( A \). Moreover, in this case, \( M \) is also a \( \mathbb{K} \)-vector space (by restriction of scalar along \( \phi : \mathbb{K} \rightarrow A \)).

Definitions 1.2.4. Let \( M \) be an \( R \)-module.

1. A subgroup \( N \leq M \) that is stable under \( r \star - \), \( \forall r \in R \) is called a submodule of \( M \);  
2. A submodule of \( R \) \( R \) is a left ideal of \( R \), whereas a submodule of \( R \) \( R \) is a right ideal;  
3. Let \( m_1, \ldots, m_n \in M \). An \( R \)-linear combination of those elements is an element of \( M \) of the form \( \sum_{i=1}^{n} r_i m_i \), for some \( r_i \in R \);  
4. The module \( M \) is said finitely generated if there exist a finite subset \( X \subseteq M \) such that every element of \( M \) is a \( R \)-linear combination of elements in \( X \);  
5. If \( N, L \leq M \), we define their sum by  
\[ N + L = \{ n + l \mid n \in N, l \in L \} \; \] 

The sum is direct if \( N \cap L = 0 \), and we note \( N + L = N \oplus L \);  
6. A submodule \( N \leq M \) is called a direct summand if there exists \( L \leq M \) such that \( M = N \oplus L \);

Definition 1.2.5 (Morphism of modules). Let \( M \) and \( N \) be two left \( R \)-modules. A morphism of modules (of \( R \)-linear maps) \( f : M \rightarrow N \) is a group homomorphism such that the following diagram commutes:

\[
\begin{matrix}
R \times M & \xrightarrow{\text{id} \times f} & R \times N \\
\star & & \star \\
M & \xrightarrow{f} & N \\
\end{matrix}
\]

We denote by \( \text{Hom}_R(M, N) \) the set of all \( R \)-linear maps from \( M \) to \( N \). The definition of morphism of right \( R \)-modules is obtained similarly.

1.2.2 Quotients

Let \( M \) be a \( R \)-module, and \( L \subseteq M \) be a submodule. Then \( L \) is a normal subgroup of \( M \), and the quotient group \( M/L \) is defined. Also, there is a quotient homomorphism \( \pi : M \rightarrow M/L \). Since \( L \) is a submodule, \( M/L \) acquires a structure of \( R \)-module:

\[ r \star \bar{m} = \bar{r \star m}, \quad \forall m \in M, r \in R. \]

Moreover : \( \pi : M \rightarrow M/L \) becomes a morphism of \( R \)-modules.
Theorem 1.2.6 (Universal property of the quotient module). For any morphism of $R$-modules $\phi : M \to N$ such that $\phi(L) = \{0\}$, there exists a unique homomorphism of $R$-modules $\hat{\phi} : M/L \to N$, such that the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & N \\
\pi \downarrow & & \downarrow \exists \hat{\phi} \\
M/L & & \\
\end{array}
\]

Proof. Follows easily from the universal property of the quotient group. \qed

Exercise 1.2.7. $\pi : M \to M/L$ induces a bijection:

\[
\{X \text{ submodule of } M \mid L \subseteq X\} \mapsto \{Y \text{ submodule of } M/L\}.
\]

Theorem 1.2.8 (Isomorphism theorems). 1. Let $\phi : M \to N$ be a homomorphism of $R$-modules. Then $\overline{\phi} : M/\ker \phi \cong \operatorname{im} \phi$ is an isomorphism.

2. Let $N$ and $L$ be submodules of $M$. Then the inclusion $L \hookrightarrow L + N$ induces an isomorphism of $R$-modules $L/(L \cap N) \cong (L + N)/N$.

3. Let $L$ and $N$ be submodules of $M$ such that $L \subseteq M$. Then we have an isomorphism $M/N \cong (M/L)/(N/L)$.

1.2.3 Exact sequences

Definition 1.2.9 (Exact sequence). A sequence $L \xrightarrow{\phi} M \xrightarrow{\psi} N$ of two homomorphism of $R$-modules is said exact if $\operatorname{im} \phi = \ker \psi$.

Properties 1.2.10. 1. $0 \to M \xrightarrow{\psi} N$ is exact if and only if $\psi$ is injective.
2. $L \overset{\phi}{\rightarrow} M \rightarrow 0$ is exact if and only if $\phi$ is surjective.

3. Let $L$ be a submodule of $M$, then the following sequence is exact:
   $$0 \rightarrow L \leftarrow M \overset{\pi}{\rightarrow} M/L \rightarrow 0.$$

4. More generally, a sequence $0 \rightarrow L \overset{\phi}{\rightarrow} M \overset{\psi}{\rightarrow} L \rightarrow 0$ is called a short exact sequence if it is exact for every two consecutive pair of homomorphism:
   - $\phi$ is injective,
   - $\text{im} \phi = \ker \psi$, and so $\psi$ induced an isomorphism $M/\text{im} \phi \overset{\cong}{\rightarrow} N$,
   - $\psi$ is surjective.

**Definition 1.2.11.** 1. A section of a homomorphism of $R$-modules $\phi : M \rightarrow N$ is a homomorphism $\sigma : N \rightarrow M$ such that $\phi \sigma = \text{id}_N$. In that case, $\phi$ is surjective and $\sigma$ is injective.

2. A retraction of $\phi$ is a homomorphism $\rho : N \rightarrow M$ such that $\rho \phi = \text{id}_M$. In that case, $\phi$ is injective, and $\rho$ is surjective.

**Proposition 1.2.12.** Let $0 \rightarrow L \overset{\alpha}{\rightarrow} M \overset{\pi}{\rightarrow} L \rightarrow 0$ be a short exact sequence of $R$-modules. The following are equivalent:

1. $\pi$ has a section,
2. $\alpha$ has a retraction,
3. $\text{im} \alpha$ is a direct summand of $M$.

**Proof.**

1. $\implies$ 2. Assume that $\pi$ has a section $\sigma : N \rightarrow M$, and define $\varepsilon = \text{id}_M - \sigma \pi : M \rightarrow M$. Clearly, $\text{im} \varepsilon \subseteq \ker \pi = \text{im} \alpha$. Using $\alpha^{-1} : \text{im} \alpha \rightarrow L$, we obtain $\rho = \alpha^{-1} \varepsilon : M \rightarrow L$. Then
   \[
   \rho \alpha = \alpha^{-1} \varepsilon \alpha = \alpha^{-1} (\alpha - \sigma \pi \alpha) = 0,
   \]
   hence $\rho = \text{id}_L$.

2. $\implies$ 3. Assume that $\alpha$ has a retraction $\rho : M \rightarrow L$. Let $Q = \ker \rho$. Then $M = Q \oplus \text{im} \alpha$. Indeed, take $m \in M$ and write $m = \alpha \rho(m) + (m - \alpha \rho(m))$. So $M = Q + \text{im} \alpha$. Let $m \in Q \cap \text{im} \alpha$. Then $\exists l \in L$ such that $\alpha(l) = m$, and so $l = \rho \alpha(l) = \rho(m) = 0$. So $l = 0$, and $m = 0$. Hence $Q \cap \text{im} \alpha = \{0\}$. 
3. \[ \implies \] 1. Assume that \( M = Q \oplus \text{im} \alpha \). We first prove that \( \pi|_Q : Q \rightarrow L \) is an isomorphism. \( \ker \pi|_Q = Q \cap \ker \pi = Q \cap \text{im} \alpha = \{0\} \). Let \( n \in N \). Since \( \pi \) is surjective, \( \exists m \in M \) such that \( \pi(m) = n \). Then \( m = q + \alpha(l) \), for some \( q \in Q \) and \( l \in L \). However, \( \pi(q + \alpha(l)) = \pi(q) \), and so \( \pi|_Q \) is surjective. So it is an isomorphism. Finally, \( \sigma = \pi|_Q^{-1} : N \rightarrow M \) is a section of \( \pi \).

\[ \square \]

**Definition 1.2.13** (Split exact sequence). Where one (and hence all) of the above condition is satisfied, the given short exact sequence is said *split*.

**1.2.4 Free modules**

**Definition 1.2.14** (Free module). An \( R \)-module \( F \) is called *free* if it has a basis, i.e. a \( R \)-generating \( R \)-linearly independent subset \( B \subseteq F \).

**Lemma 1.2.15.** Let \( F \) be an \( R \)-module. Then \( F \) is finitely generated free if and only if \( \exists n \in \mathbb{N} \) such that \( F \cong R^n \).

**Proposition 1.2.16.**

1. Any \( R \)-module is isomorphic to a quotient of a free module.

2. Any finitely generated \( R \)-module is isomorphic to a quotient of a finitely generated free module.

**Proof.** Let \( M \) be an \( R \)-module, and \( G = \{g_i \mid i \in I\} \) be a set of generators of \( M \) (such a set exists : take the whole \( M \) for instance). Let \( F \) be a free module with basis \( B = \{b_i \mid i \in I\} \), and define \( \phi : F \rightarrow M \)

\[
\phi : F \rightarrow M \\
\phi(b_i) = g_i,
\]

extended by \( R \)-linearity. Then \( M \cong F/\ker \phi \).

**1.3 Projective modules**

**Proposition 1.3.1.** Let \( R \) be a ring, and let \( P \) be a left \( R \)-module. The following are equivalent:

1. \( P \) is isomorphic the a direct summand of a free module;

2. for every surjective homomorphism \( \pi \) and any \( \phi \), there exists a lift as follows:

\[
\begin{array}{c}
P \\
\pi \downarrow \\
L \xrightarrow{} M;
\end{array}
\]
3. for any surjective homomorphism \( \phi : M \to P \), there exists a section \( \sigma : P \to M \); 

4. any short exact sequence of the form \( 0 \to L \to M \to P \to 0 \) splits.

Proof.

1. \( \implies \) 2. by assumption, \( P \cong P' \), and \( P' \oplus Q = F \) free. Let \( \eta : P \xrightarrow{\cong} P' \xrightarrow{\rho} F \), and \( \rho \xrightarrow{\sigma} P' \xrightarrow{\cong} P \).

\[
\begin{array}{c}
F \\
\alpha \downarrow \\
\rho \\
\downarrow \\
P \xrightarrow{\phi} M \\
\pi \\
N
\end{array}
\]

\( F \) has a basis \( G = \{g_i \mid i \in I\} \). Let \( m_i \) be a preimage of \( \phi \rho(g_i) \) by \( \pi \), i.e. \( \pi(m_i) = \phi \rho(g_i) \). As \( F \) is free, there exists a map \( \alpha : F \to M \) such that \( \alpha(g_i) = m_i \). Define \( \tilde{\phi} = \alpha \eta : P \to M \). Then \( \pi \tilde{\phi} = \pi \alpha \eta = \phi \rho \eta = \phi \).

2. \( \implies \) 3. Consider

\[
\begin{array}{c}
\exists \sigma \\
\downarrow \\
P \\
\pi \downarrow \\
M \\
\xrightarrow{\pi} P
\end{array}
\]

3. \( \implies \) 4. The homomorphism \( M \to P \) in the short exact sequence is surjective, and so admits a section.

4. \( \implies \) 1. Recall that \( P \) is a quotient of a free module. So we have an exact sequence \( 0 \to \ker \pi \to F \xrightarrow{\pi} P \to 0 \). It splits by assumption, and so \( F \cong P \oplus \ker \pi \).

\[\square\]

**Definition 1.3.2** (Projective module). A module satisfying one (and hence all) of the above condition is called projective.
Chapter 2

Semisimplicity

2.1 Definitions

Definition 2.1.1 (Simple module). Let $R$ be a ring, and $S$ be a left $R$-module. It is said simple if it is not trivial, and if it has no submodule other than 0 and $S$.

Lemma 2.1.2. $S$ is simple if and only if $S$ is isomorphic to $R/I$ (as modules), where $I$ is a maximal left ideal of $R$.

Proof. $\iff$ We have $S \cong R/I$. A submodule of $S$ corresponds to a submodule $I \leq M \leq R$. Hence $S$ is simple.

$\implies$ Let $S$ be simple. In particular, $S \neq 0$ and $\exists s \in S \setminus \{0\}$. Since $R$ is a free $R$-module with basis 1, there exists a homomorphism $\pi : R \to S$ such that $\pi(1) = s$. We have $0 \neq \text{im } \pi \leq S$, and as $S$ is simple, $\text{im } \pi = S$. By the first isomorphism theorem, $S \cong R/\ker \pi$. By simplicity of $S$, $\ker \pi$ is maximal.

Lemma 2.1.3 (Schur, general case). 1. Let $S$ be a simple left $R$-module. Then $\text{end}_R S$ as an $R$-module is a division ring.

2. If $S$ and $T$ are two nonisomorphic simple $R$-modules, then $\text{Hom}_R(S, T) = 0$.

Proof. 1. Let $\phi \in \text{end}_R S$. Remark that $\ker \phi, \text{im } \phi \leq S$, and so either

(a) $\text{im } \phi = 0$, $\ker \phi = S$, in which case $\phi = 0$;

(b) $\ker \phi = 0$, $\text{im } \phi = S$, in which case $\phi$ is an automorphism, hence invertible.

2. Similarly, a homomorphism $\phi : S \to T$ is either 0 or an isomorphism.

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Lemma 2.1.4 (The real lemma of Schur). Let $A$ be a finite dimensional $\mathbb{K}$-algebra, and let $S$ be a simple $A$-module.

1. $K \hookrightarrow \text{end}_A S \hookrightarrow \text{end}_\mathbb{K} S = \text{Mat}_n(\mathbb{K})$, where $n = \dim_\mathbb{K} S$;

2. if $\mathbb{K}$ is algebraically closed, then $K \cong \text{end}_A S$.

Proof. 1. If $S$ is simple, we have $S \cong A/I$, for some (maximal) left ideal $I$, and so $S$ is finitely generated. So $S$ is finite dimensional (cf exercise 1, sheet 2) as a $\mathbb{K}$-vector space. Therefore, $\text{end}_\mathbb{K} S \leq \text{Mat}_n(\mathbb{K})$, where $n = \dim_\mathbb{K} S$. Obviously, $\text{end}_A S$ is a subalgebra of $\text{end}_\mathbb{K} S$, and we have an embedding $K \hookrightarrow \text{end}_A S : \lambda \mapsto \lambda \text{id}_S$.

2. Assume $\mathbb{K}$ algebraically closed, and let $\phi \in \text{end}_A S$. Let $\mathbb{K}[X]$ be the polynomial ring in one variable $X$, and define $\pi : \mathbb{K}[X] \rightarrow \mathbb{K}[\phi] \leq \text{end}_A S : X \mapsto \phi$. Then $\pi$ is a $\mathbb{K}$-algebra map. By the general case of Schur lemma, we know that $\text{end}_A S$ is a division algebra, and so $\rho \psi \neq 0$ whenever $\rho, \psi \in \text{end}_A S$. Therefore, the commutative subalgebra $\mathbb{K}[\phi]$ is a domain. Since $\mathbb{K}[X]$ is a PID, $\ker \pi$ is generated by a single polynomial $f$ which can be chosen monic ($\pi$ can’t be injective, because $\mathbb{K}[X]$ is infinite dimensional, and $\dim \text{end}_A S \leq n^2$), i.e. $f$ is the minimal polynomial of $\phi$. So $\mathbb{K}[\phi] \cong \mathbb{K}[X]/(f)$. But this is a domain, so $f$ is irreducible. Since $\mathbb{K}$ is algebraically closed, $f(X) = X - \lambda$, for a $\lambda \in \mathbb{K}$. It follows that $\mathbb{K}[\phi] \cong \mathbb{K}[X]/(X - \lambda) \cong \mathbb{K}$, and so $\phi = \lambda \text{id}_S$. Hence, the map $\mathbb{K} \rightarrow \text{end}_A S : \lambda \mapsto \lambda \text{id}_S$ is an isomorphism. \hfill $\square$

Proposition 2.1.5. Let $A$ be a finite dimensional $\mathbb{K}$-algebra, and let $M$ a be finitely generated left $A$-module. The following are equivalent:

1. $M$ is a sum of simple submodules, i.e. $M = \sum_{i \in I} S_i$, where $S_i \leq M$ is simple;

2. $M$ is a direct sum of finitely many simple submodules, i.e. $M = \bigoplus_{j \in J} S_j$, where $S_j \leq M$ is simple, and $J$ finite;

3. every submodule $N \leq M$ is a direct summand of $M$.

Proof.

1. $\implies$ 2. Define $X = \{ J \subseteq I \mid \sum_{j \in J} S_j \text{ is direct} \}$. Such a $J \in X$ must be finite, as $\infty > \dim_\mathbb{K} M \geq \dim_\mathbb{K} \sum_{j \in J} S_j = \dim_\mathbb{K} \bigoplus_{j \in J} S_j = \sum_{j \in J} \dim_\mathbb{K} S_j \geq \# J$. Take a maximal element $J$ in $X$. Then we claim that $S_i \subseteq \bigoplus_{j \in J} S_j$, $\forall i \in I$. Assume $i \notin J$. If $S_i \not\subseteq \bigoplus_{j \in J} S_j$, then $S_i \cap \bigoplus_{j \in J} S_j$ is a proper submodule of $S_i$, and so it is 0. So the sum $S_i + \bigoplus_{j \in J} S_j$ is direct, which is absurd by maximality of $J$. So $M = \sum_{i \in I} S_i \subseteq \bigoplus_{j \in J} S_j$, and finally, $M = \bigoplus_{j \in J} S_j$. \hfill $\square$
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2. $\implies$ 3. We have $M = \bigoplus_{j \in J} S_j$, and let $N \leq M$. Suppose $N \neq M$, and define $Y = \{L \subseteq J \mid N + \bigoplus_{l \in L} S_l \text{ is direct}\}$. Let $L$ be maximal in $Y$. We claim that $N' = \bigoplus_{l \in L} S_l$ is not direct by maximality of $L$. So $N \cap N' \neq 0$, so it is $S_j$ by simplicity. For every $j \in J$, $S_j \subseteq N + \bigoplus_{l \in L} S_l$, and so $N + \bigoplus_{l \in L} S_l = M$.

3. $\implies$ 1. Let $T = \sum_{S \subseteq M, S \text{ simple}} S$. It is a submodule of $M$, and so by assumption, $M = T \oplus U$, for some $U \leq M$. If $U \neq 0$, then we choose a nonzero submodule $V \leq U$ of minimal $K$-dimension. Then $V$ must be simple by minimality, and then we get a direct sum $T \oplus V \leq T \oplus U = M$. On the other hand, $V \leq T$ by definition of $T$. So $V = U = 0$, and $T = M$.

\[\square\]

Definition 2.1.6. An $A$-module $M$ satisfying one (and hence all) of the above condition is said semisimple.

Remark 2.1.7. The results of proposition 2.1.5 holds for any ring $R$. The proof uses Zorn’s lemma.

Definition 2.1.8. 1. A finitely generated $A$-module $M$ is called semisimple isotypic if $M$ is the direct sum of isomorphic simple modules. We have $M \cong S^{\oplus k}$, for $S$ a simple module, and we call $S$ its type. Lemma 2.1.9 shows that this definition is well founded.

2. If $M$ is semisimple, then we can group all isomorphic simple summands in its decomposition as a direct sum of simple modules, and obtain the following:

\[
M = (S_{1,1} \oplus \cdots \oplus S_{1,n_1}) \oplus \cdots \oplus (S_{m,1} \oplus \cdots \oplus S_{m,n_m})
\]

where $S_{i,j} \neq S_{k,l}$ whenever $i \neq k$. Then $S_i = \bigoplus_{j=1}^{n_i} S_{i,j}$ is called an isotypic component of $M$, of type $S_{i,1} \cong \cdots \cong S_{i,n_i}$.

Lemma 2.1.9. 1. Any simple submodule of a semisimple isotypic module $N$ of type $S$ is isomorphic to $S$.

2. If $M = \bigoplus_S M_S$ is a decomposition of $M$ into isotypic components, with $M_S$ of type $S$, then any simple submodule of $M$ isomorphic to $S$ is contained in $M_S$.

3. $M_S$ only depends on the type $S$, not on the chosen simple decomposition of $M$. Explicitly, $M_S = \bigoplus_{T \leq M, T \cong S} T$. 


Proof. 1. \( N = \bigoplus_{i=1}^{n} S_i \), where \( S_i \cong S \). Let \( T \leq N \) be simple. Then \( \text{proj}_{S_i} T \) is not always 0, otherwise we would have \( T = 0 \). Take \( j \) such that \( 0 \neq \text{proj}_{S_j} T \leq S_j \). Because \( S_j \) is simple, we have \( \text{proj}_{S_j} T = S_j \). Moreover, since \( T \) is simple, \( \ker(\text{proj}_{S_j} : T \rightarrow S_j) = 0 \), and so \( T \cong S_j \).

2. Let \( T \leq M \) be a simple submodule isomorphic to \( S \). There exists \( U \leq M \) simple such that \( \text{proj}_{M_U} T \neq 0 \). However, \( \text{proj}_{M_U} T \) is a simple submodule of \( M_U \) isomorphic to \( T \). So by the previous point, \( U \cong T \cong S \). The same reasoning applies to show that \( \text{proj}_{M_V} T = 0 \), whenever \( V \neq S \). So \( T \leq M_S \).

3. Obvious from previous points. \( \square \)

Example 2.1.10. Let \( A = \mathbb{K} \). A theorem in linear algebra states that every \( A \)-module (\( \mathbb{K} \)-vector space) has a basis. So every \( A \)-module is semisimple isotypic of type \( \mathbb{K} \).

Definition 2.1.11. Let \( A \) be a finite dimensional \( \mathbb{K} \)-algebra.

1. \( A \) is called simple algebra if \( A \) is a semisimple module.

2. \( A \) is called simple if \( A \) is semisimple isotypic.

Proposition 2.1.12. The following are equivalent:

1. \( A \) is semisimple;

2. every finitely generated left \( A \)-module is projective;

3. every finitely generated left \( A \)-module is semisimple.

Proof.

1. \( \implies \) 2. Let \( M \) be a finitely generated left \( A \)-module. Then \( \exists n \in \mathbb{N} \) such that \( M \cong F/N \), where \( F = A^{\oplus n} \) is free, and \( N \leq F \). Moreover, \( F \) is semisimple, and so \( N \) is a direct summand of \( F \), i.e. \( F = N \oplus Q \), for some submodule \( Q \). So \( M \cong Q \) is projective.

2. \( \implies \) 3. Let \( N \) be a submodule of \( M \). We have a short exact sequence \( 0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0 \). By assumption, \( M/N \) is projective, so the previous exact sequence splits, and \( M \cong N \oplus M/N \). So \( N \) is a direct summand of \( M \), and \( M \) is semisimple.

3. \( \implies \) 1. Remark that \( A \) is a finitely generated \( A \)-module. \( \square \)

Exercise 2.1.13. If \( A \) is semisimple, let \( A \cong \bigoplus_{i=1}^{r} M_{S_i} \) be its isotypic decomposition. Prove that any simple \( A \)-module is isomorphic to one \( S_i \).
2.2. The Wedderburn classifications theorem

Example 2.2.1. Let $D$ be a finite dimensional division $\mathbb{K}$-algebra. Then

$$D^{\oplus n} = \left\{ \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \mid d_1, \ldots, d_n \in D \right\}$$

is a left $M_n(D)$-module. Then $M_n(D)D^{\oplus n}$ is simple because if $s \in D^{\oplus n}, s \neq 0$, then $s_i \neq 0$ for some $i$, then

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_i & 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} s_1^{-1} \\ \vdots \\ s_n^{-1} \\ 1 \end{pmatrix} = \begin{pmatrix} t_1 \\ \vdots \\ t_n \\ 1 \end{pmatrix},$$

and so $M_n(D)s = D^{\oplus n}$.

Now, $M_n(D)$ is a simple $\mathbb{K}$-algebra because

$$M_n(D) = \begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & * & \cdots & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \cdots & 0 & * \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix} \cong D^{\oplus n} \oplus D^{\oplus n} \oplus \cdots \oplus D^{\oplus n} \cong D^{\oplus n}.$$

Exercise 2.2.2. Any product $\prod_i M_n(D_i)$ is semisimple, where $D_i$ is a division $\mathbb{K}$-algebra.

Lemma 2.2.3. Let $P$ and $Q$ be two left $A$-modules, and suppose that $P = \bigoplus_{i=1}^p X_i$, $Q = \bigoplus_{i=1}^q Y_i$. Let $\varepsilon_j : X_j \rightarrow P$ be the inclusions, and $\pi_i : Q \rightarrow Y_i$ be the projections. Denote $\text{Hom} = \text{Hom}_A$.

1. Define

$$M = \begin{pmatrix} \text{Hom}(X_1, Y_1) & \cdots & \text{Hom}(X_p, Y_1) \\ \vdots & \ddots & \vdots \\ \text{Hom}(X_1, Y_q) & \cdots & \text{Hom}(X_p, Y_q) \end{pmatrix} \cong \bigoplus_{i,j} \text{Hom}(X_i, Y_j).$$

Then $M \cong \text{Hom}(P, Q)$ as $A$-modules.

2. If $P = Q$, $p = q$, $X_i = Y_i$, then the isomorphism of the previous point is an isomorphism of $\mathbb{K}$-algebras : $\text{end} \ P \cong M$ as rings, where $M$ is endowed with the usual matrix multiplication.
CHAPTER 2. SEMISIMPlicity

Proof. 1. Remark that

$$\phi : \text{Hom}(P, Q) \to M$$

$$f \mapsto \begin{pmatrix} \pi_1 f \varepsilon_1 & \cdots & \pi_1 f \varepsilon_p \\ \vdots & \ddots & \vdots \\ \pi_q f \varepsilon_1 & \cdots & \pi_q f \varepsilon_p \end{pmatrix}$$

is an isomorphism with inverse

$$\psi : M \to \text{Hom}(P, Q)$$

$$\begin{pmatrix} b_{1,1} & \cdots & b_{p,1} \\ \vdots & \ddots & \vdots \\ b_{1,q} & \cdots & b_{p,q} \end{pmatrix} \mapsto \psi((b_{i,j})_{i,j}),$$

$$\psi((b_{i,j})_{i,j}) : P \to Q$$

$$(x_1, \ldots, x_p) \mapsto \left( \sum_{j=1}^q b_{1,j}(x_j), \ldots, \sum_{j=1}^q b_{p,j}(x_j) \right).$$

2. The fact that $P = Q$, $p = q$, $X_i = Y_i$ make the matrix multiplication of $M$ well defined. The rest is routine verifications.

\[\square\]

Corollary 2.2.4. 1. Let $S$ and $T$ be two nonisomorphic simple $A$-modules. Then $\text{Hom}_A(S^\oplus p, T^\oplus q) = 0$. 

2. Let $S$ be an $A$-module (not necessarily simple). Then $\text{end}_A(S^\oplus p) \cong M_p(\text{end}_A S)$.

Proof. 1. By previous lemma, we have $\text{Hom}_A(S^\oplus p, T^\oplus q) \cong M_{q \times p}(\text{Hom}_A(S, T)) = 0$.

2. By previous lemma.

\[\square\]

Theorem 2.2.5 (Wedderburn). Let $A$ be a finite dimensional $\mathbb{K}$-algebra. If $A$ is semisimple, then $A \cong \prod_{i=1}^k M_{n_i}(D_i)$, where $D_i$ is a division $\mathbb{K}$-algebra, $n_i \geq 1$. Moreover, $D_i^{\text{op}} \cong \text{end}_A S_i$, where $S_i$ is a simple $A$-module.

Proof. We have that $A$ is semisimple, and let $A \cong \bigoplus_{i=1}^k S_i^{\oplus n_i}$ be its isotypic
decomposition. By the lemma,
\[
\begin{pmatrix}
\text{end}_A S_1^\oplus n_1 & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & \text{end}_A S_r^\oplus n_r
\end{pmatrix}
\]
\[
\cong \prod_{i=1}^r \text{end}_A S_i^\oplus n_i
\]
\[
\cong \prod_{i=1}^r M_{n_i}(\text{end}_A S_i).
\]

By Schur’s lemma, \(\text{end}_A S_i\) is a division algebra. But we have an isomorphism of algebras \(A^\text{op} \cong \text{end}_A A\). Indeed:
\[
\begin{align*}
A^\text{op} & \to \text{end}_A A \\
 b & \mapsto m_b, \\
\text{end}_A A & \to A^\text{op} \\
 f & \mapsto f(1),
\end{align*}
\]
are mutually inverse. Hence,
\[
A \cong (\text{end}_A A)^\text{op} \cong \left(\prod_{i=1}^r M_{n_i}(\text{end}_A S_i)^\text{op}\right) = \prod_{i=1}^r M_{n_i} \left(\text{end}_A S_i\right)^\text{op}.
\]
The isomorphism \(M_n(R)^\text{op} \to M_n(R^\text{op})\) is given by transposition. \(\Box\)

**Corollary 2.2.6.** Suppose that \(\mathbb{K}\) is algebraically closed. If \(A\) is a semisimple \(\mathbb{K}\)-algebra, then \(A \cong \prod_{i=1}^r M_{n_i}(\mathbb{K})\).

**Proof.** In this case, \(\text{end}_A S_i \cong \mathbb{K}\) by Schur’s lemma, and so \(D_i = M_{n_i}(\mathbb{K})^\text{op} = M_{n_i}(\mathbb{K})\) because \(M_{n_i}(\mathbb{K})\) is commutative. \(\Box\)

**Corollary 2.2.7.** If \(A\) is a simple \(\mathbb{K}\)-algebra, then \(A \cong M_n(D)\), where \(D = (\text{end}_A S)^\text{op}\), where \(S\) is the unique simple \(A\)-module up to isomorphism.

**Theorem 2.2.8** (Maschke). Take \(G\) a finite group, and \(\mathbb{K}\) a field. Then \(\mathbb{K}G\) is semisimple if and only if \(\text{char } \mathbb{K} \nmid |G|\), i.e. \(|G| \neq 0\) in \(\mathbb{K}\).

**Proof.** \(\implies\) Suppose that \(\mathbb{K}G\) is semisimple. Consider
\[
\varepsilon : \mathbb{K}G \to \mathbb{K} \\
\sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g.
\]
This makes \(\mathbb{K}\) into a \(\mathbb{K}G\) module (with trivial \(G\)-action), and \(\varepsilon\) is \(\mathbb{K}G\)-linear surjective. By semisimplicity, \(\varepsilon\) splits (as \(\mathbb{K}\) is projective). Let
\[ \sigma : \mathbb{K} \to \mathbb{K}G \text{ a } \mathbb{K}G\text{-linear section of } \varepsilon. \] Put \( \sigma(1) = \sum_{g \in G} \lambda_g g \). Take \( h \in G \), then \( h\sigma(1) = \sigma(h1) = \sigma(1) \), and so
\[
\sum_{g \in G} \lambda_g hg = \sum_{g \in G} \lambda_g g, \quad \forall h \in G.
\]
Hence all \( \lambda_g \)'s are equal. Write \( \sigma(1) = \lambda \sum_{g \in G} g \). Then
\[
1 = \varepsilon \sigma(1) = \lambda \varepsilon \left( \sum_{g \in G} g \right) = \lambda |G|,
\]
and so \( |G| \neq 0 \) in \( \mathbb{K} \).

\( \Leftarrow \Rightarrow \) Suppose \( \text{char } \mathbb{K} \nmid |G| \). We prove that every \( \mathbb{K}G \)-module \( P \) is projective. Let \( 0 \to N \xrightarrow{\delta} M \xrightarrow{\pi} P \to 0 \) be a short exact sequence. The sequence splits as a sequence of \( \mathbb{K} \)-vector spaces and so there exists \( \sigma : P \to M \) a \( \mathbb{K} \)-linear section of \( \pi \). Define \( \tau = \frac{1}{|G|} \sum_{g \in G} g\sigma(g^{-1} \cdot -) : P \to M \).
It is easy to see that \( \tau \) is a \( \mathbb{K}G \)-linear section of \( \pi \), so the sequence splits, and \( P \) is projective.

\[ \square \]

**Exercise 2.2.9.** Let \( A \) be a finite dimensional semisimple \( \mathbb{K} \)-algebra, and \( S_i \) be a simple \( A \)-module. Prove that the multiplicity \( n_i \) of \( S_i \) in the semisimple decomposition of \( A \) is \( \text{dim}_{D_i} S_i \), where \( D_i = \text{end}_A S_i \).
Chapter 3

The Jacobson radical

3.1 Definition

Definition 3.1.1 (Composition series). Let $M$ be a an $R$-module, where $R$ is a ring. A *composition serie* of $M$ is a sequence of submodules

$$0 = M_k < M_{k-1} < \cdots < M_0 = M$$

such that every quotient $M_i/M_{i-1}$ is simple. Such a quotient is called a *composition factor*.

Lemma 3.1.2. If $A$ is a finite dimensional $K$-algebra, and $M$ a finitely generated $A$-module, then it admits a composition serie.

Proof. Recall that $M$ is finite dimensional. Take a proper submodule $N$ of maximal dimension. Then $M/N$ is simple. Repeat with $N$. The process eventually stops as $M$ is finite dimensional.

Theorem 3.1.3 (Jordan–Holder). Let $A$ be a finite dimensional $K$-algebra, and $M$ a finitely generated left $A$-module. Any two composition series of $M$ have the same length and isomorphic quotients up to permutation. Explicitly, if

$$0 = M_r < M_{r-1} < \cdots < M_0 = M, \quad 0 = N_s < N_{s-1} < \cdots < N_0 = M$$

are two composition series, then $r = s$, and there exists $\sigma \in \mathfrak{S}_r$ such that $N_{i-1}/N_i \cong M_{\sigma(i)-1}/M_{\sigma(i)}$.

Proof. By induction on $\dim_K M$. If $\dim_K M = 1$, then $M$ is simple, and the result is obvious. If $M_1 = N_1$, then the result follows by induction hypothesis. Suppose now that $M_1 \neq N_1$. Then $M_1 + N_1 = M$ by maximality
Chapter 3. The Jacobson Radical

of $M_1$ and $N_1$.

\[ \begin{array}{ccc}
M & & N_1 \\
M_1 & & \downarrow & \downarrow \\
M_2 & M_1 \cap N_1 & N_2 \\
\vdots & \vdots & \vdots \\
& Q_1 & \\
\end{array} \]

Let $0 = Q_k < \cdots < Q_0 = M_1 \cap N_1$ be a composition series. The two composition series of $M_1$ give (by induction) $r - 1 = k + 1$, and isomorphic quotients. Therefore, the composition factor of $M_1$ are, up to isomorphism, $\{\text{composition factors of } M_1 \cap N_1 \} \cup \{M_1/M_1 \cap N_1\}$. Similarly for $N_1$:

1. The composition factors are, up to isomorphism, $\{\text{composition factors of } M_1 \cap N_1 \} \cup \{N_1/M_1 \cap N_1\}$. Then the factors of the first series for $M$ are, up to isomorphism $\{\text{composition factors of } M_1 \cap N_1 \} \cup \{M/M_1, M_1/M_1 \cap N_1\}$. For the second series, we have $\{\text{composition factors of } M_1 \cap N_1 \} \cup \{M/N_1, N_1/M_1 \cap N_1\}$. Moreover, $M/M_1 \cong N_1/M_1 \cap N_1$, and $M/N_1 \cong M_1/M_1 \cap N_1$. So the two composition series have the same factors.

Corollary 3.1.4. There are finitely many simple $A$-modules, up to isomorphism.

Proof. It $S$ is simple, then $S \cong A/J$, for some maximal ideal $J$. Then $S$ is a composition factor of the following composition series of $0 < \cdots < J < A$. However, $A$ only has finitely many composition factors.

Definition 3.1.5 (Jacobson radical). Let $A$ be a finite dimensional $k$-algebra, and $M$ a finitely generated $A$-module.

1. The Jacobson radical of $M$, written $J(M)$, is the intersection of all maximal submodules of $M$.

2. The Jacobson radical of $A$ is $J(A) = J(AA)$, the intersection of all maximal left ideals.

Example 3.1.6. If $M$ is semisimple, then $J(M) = 0$. Indeed, $M = \bigoplus_i S_i$, and so $\bigoplus_{i \neq j} S_i$ is a maximal submodule. Hence $J(M) \leq \bigcap_j \bigoplus_{i \neq j} S_i = 0$. 

3.2 Caracterisations

Proposition 3.2.1. 1. Let $I$ be a left ideal of $A$. Then $I$ is maximal if and only if there exists $S$ a simple module, and $s \in S$, $s \neq 0$ such that $I = \text{Ann}(s)$, where $\text{Ann}(s) = \{ a \in A \mid as = 0 \}$ is the annihilator of $s$.

2. $J(A) = \bigcap_{S \text{ simple}} \text{Ann}(S)$, where $\text{Ann}(S) = \{ a \in A \mid aS = 0 \} = \bigcap_{s \in S} \text{Ann}(s)$ is the annihilator of $S$.

3. $J(A)$ is a two sided ideal.

Proof. 1. $A/I$ is simple, and so $I = \text{Ann}(1)$, for $1 \in A/I$. Conversely, if $S$ is simple, $s \neq 0$, then $As = S$, and so $A/\text{Ann}(s) \cong S$ using the first isomorphism theorem for the map $\phi : A \longrightarrow S, a \longmapsto as$.

2. We have

$$J(A) = \bigcap_{I \text{ max. left ideal}} I$$

$$= \bigcap_{S \text{ simple}} \bigcap_{0 \neq s \in S} \text{Ann}(s).$$

3. Let $a \in J(A)$, and $b \in A$. Let $S$ be a simple module. Then $abS = 0$, as $bS$ is either $S$ or $0$. Hence, $ab \in \text{Ann}(S)$, for all simple module $S$, and so $ab \in J(A)$.

Proposition 3.2.2. Let $I$ be a left ideal of $A$. Then $I \leq J(A)$ if and only if $1 + I \leq A^\times$

Proof. $\implies$ $a \in I \subseteq J(A)$, and so $a \in m$, for every maximal ideal $m$ of $A$, and so $1 + a \notin m$ (otherwise, $1 \in m$). However, an element is non left invertible if and only if it is contained in some (without loss of generality maximal) left ideal. So $1 + a$ is left invertible : $\exists u \in A$ such that $u(1 + a) = 1$. Now $u = 1 - ua$, and $-ua \in I$, as $I$ is a left ideal.

The same argument applied to $ua$ shows that $-ua$ is left invertible : $\exists v \in A$ such that $v(1 - ua) = 1$. However $1 - ua = u$, and so $vu = 1$.

Therefore $1 + a = vu(1 + a) = v$, and so $1 + a$ has right inverse $u$.

$\Longleftarrow$ Let $m$ be a maximal left ideal of $A$. If $a \notin m$, then $Aa + m = A$, because $m$ is maximal. So $ba + m = 1$, for some $b \in A$, $m \in m$. Hence $1 - ba = m \in m$, but $1 - ba \in A^\times$ as $-ba \in I$, which is absurd. Therefore $a \in m$, and this holds for every maximal left ideal $m$, so $a \in J(A)$.
Lemma 3.2.3 (中山の補題). Let $A$ be a finite dimensional $\mathbb{K}$-algebra, and $M$ a finitely generated left $A$-module. Then $J(A)M = M$, then $M = 0$.

Proof. Let $m_1, \ldots, m_r$ be a minimal set of generators of $M$. So $M = \sum_i Am_i$. Since $J(A)M = M$, the element $m_r$ can be written $m_r = \sum a_im_i$, with $a_i \in J(A)$. Therefore $a_1m_1 + \cdots + a_{r-1}m_{r-1} = (1 - a_r)m_r$. So $M = \sum_{i<r} a_im_i$, a contradiction with minimality of $\{m_i\}_{1 \leq i \leq r}$. □

Lemma 3.2.4 (中山の補題, alternate form). Let $A$ be a finite dimensional $\mathbb{K}$-algebra, $M$ a finitely generated left $A$-module, and $N \subseteq M$. If $J(A)M + N = M$, then $N = M$.

Proof. See exercise 2 from sheet 6. □

Theorem 3.2.5. Let $A$ be a finite dimensional $\mathbb{K}$-algebra. Then $J(A)$ is the smallest two sided ideal with semisimple quotient. Explicitely:

1. $A/J(A)$ is semisimple,

2. and if $A/I$ is also semisimple, with $I$ a two sided ideal, then $J(A) \leq I$.

Proof. We claim that $J(A)$ is an intersection of finitely many maximal left ideals. Let $I = \bigcap_i m_i$ be a finite intersection of maximal left ideals, with $\text{codim}_K I = \dim_K A - \dim_K I$ as large as possible (bounded by $\dim_K A$). If $m$ is a maximal left ideal, then $I \cap m$ is a finite intersection with larger codimension, and so by maximality of $\text{codim} K I$ and $m$, we have $I \cap m = I$, hence $I \leq m$. This argument applies for every left ideal $m$, and so $J(A) \leq I \leq J(A)$, which proves the claim.

1. Now, $J(A) = \bigcap_i m_i$. So the obvious homomorphism $A/J(A) \rightarrow \bigoplus_i A/m_i$ is injective. Each summand of the latter is simple, so the sum is semisimple. Therefore, $A/J(A)$ is isomorphic to a submodule of a semisimple module, so it is itself semisimple.

2. Let $I$ be a two sided ideal such that $A/I = \bigoplus_i S_i$ is semisimple. Note that $\text{Ann}(A/I) = I$, and so $I \leq \text{Ann}(S_i)$. Hence, $J(A) \leq I$. □

Corollary 3.2.6. $A$ is semisimple if and only of $J(A) = 0$.

Example 3.2.7. $\mathbb{Z}$ is not semisimple, but $J(\mathbb{Z}) = \bigcap_{\text{prime}} \mathbb{Z}_p = 0$.

Theorem 3.2.8. Let $A$ be a finite dimensional $\mathbb{K}$-algebra. Then $J(A)$ is the largest two sided nilpotent ideal. Explicitely

1. $J(A)$ is nilpotent,
2. if $I$ is a two sided nilpotent ideal, then $I \leq J(A)$.

**Proof.** 1. Consider a composition series $0 = M_r < \cdots < M_0 = A$. Then every $M_i$ is a left ideal. Since $M_i/M_{i+1}$ is simple, then $J(A)M_i/M_{i+1} = 0$, hence $J(A)M_i \leq M_{i+1}$. Therefore

\[
J(A)^r A = J(A)^r M_0 \leq J(A)^{r-1} M_1 \leq \cdots \leq M_r = 0.
\]

2. Let $S$ be a simple $A$-module. Then $IS \leq S$. If $IS = S$, then $I^n S = S$, and for $n$ large enough, $I^n = 0$, so $S = 0$, a contradiction. So $IS = 0$. So $I \leq \bigcap_{S \text{ simple}} \text{Ann}(S) = J(A)$.

**Example 3.2.9.** Take $A = \mathbb{K}[t]/(t^n)$, then $J(A) = (t)$. It is obviously nilpotent, and $A/J(A) \cong \mathbb{K}$ is semisimple.

**Proposition 3.2.10.** Let $M$ be a finitely generated $A$-module, where $A$ is a finite dimensional $\mathbb{K}$-algebra.

1. If $N \leq M$ such that $M/N$ is semisimple, then $J(M) \leq N$.

2. $J(M) = J(A)M$.

3. $M/J(M)$ is semisimple.

**Proof.** 1. If $M/N$ is semisimple, then $M/N = \bigoplus_i S_i$, and so $J(M/N) = 0$ (see example 3.1.6). So $\bigcap_{N \leq L \text{ max. submod.}} L = N$. So $J(M) \leq N$.

2. If $L \leq M$ is a maximal submodule, then $J(A)M/L = 0$, as $M/L$ is simple. So $J(A)M \leq L$. Therefore $J(A)M \leq \bigcap_{L \text{ max. submod.}} L = J(M)$. Conversely, $M/(J(A)M)$ is a $A/J(A)$-module. The latter algebra is semisimple, and so is $M/(J(A)M) = \bigoplus_i S_i$, where $S_i$ is a simple $A/J(A)$-module, hence a simple $A$-module. So $M/(J(A)M)$ is a semisimple $A$-module. By previous point, $J(M) \leq J(A)M$.

3. Already proved.

### 3.3 Local rings

**Definition 3.3.1** (Local ring). Let $R$ be a ring. Then $R$ is local if $R$ has a unique maximal left ideal.

**Lemma 3.3.2.** Let $R$ be a local ring, and let $J$ be its unique maximal left ideal. Then:

---

\(^1\)That trick wouldn’t work for an arbitrary restriction of scalars
CHAPTER 3. THE JACOBSON RADICAL

1. $J$ is two sided.

2. $J = R \setminus R^\times$.

3. $R/J$ is a division ring.

Proof.
1. If $b \in R \setminus \{0\}$, then Ann$(b)$ is a proper left ideal, as it doesn’t contain 1, so Ann$(b) \leq J$. If $Jb \not\subseteq J$, then we would get $Jb = R$. So $b$ is left invertible, by an element $a \in J$. Then $(1 - ba)b = b - bab = 0$. So $1 - ba \in$ Ann$(b) \leq J$. But $a \in J$, so $1 = 1 - ba + ba \in J$, which is absurd.

2. If $r \in R \setminus R^\times$, then $r$ is contained in a proper ideal, namely $(r)$. So it is contained in a maximal ideal (using Zorn’s lemma), which necessarily is $J$. So $r \in J$, and $R \setminus R^\times \subseteq J$. Conversely, $J$ cannot have any left unit (as it is a left ideal), nor right unit (as it is a right ideal), hence the equality.

3. Obvious. 

Lemma 3.3.3. Let $R$ be a ring. Suppose that every element of $R$ is either invertible or nilpotent. Then $R$ is local.

Proof. Recall that an invertible element cannot be nilpotent. Take $J$ the set of all nilpotent elements of $R$. Let $a \in J$ and $b \in R$. The $ba$ is a zero divisor : $ba \cdot a^{n-1} = ba^n = 0$, where $n$ is the least integer such that $a^n = 0$. So $ba$ is not invertible, hence nilpotent, hence $ba \in J$. Let $a_1, a_2 \in J$. Suppose $a_1 + a_2 \not\in J$, hence invertible. So $xa_1 + xa_2 = 1$, for some $x$. Note that $xa_1$ and $xa_2$ commute (as $r$ and $1 - r$ always commute, for any $r \in R$). Let $n_1$ and $n_2$ be integers such that $(xa_1)^{n_1} = 0$, and $N = n_1 + n_2$. We can use Newton’s formula (as $xa_1$ and $xa_2$ commute) :

$$1 = (xa_1 + xa_2)^N = \sum_{i=0}^{N} \binom{N}{i} (xa_1)^i (xa_2)^{N-i},$$

indeed, we necessarily have that $i \geq n_1$ or $N - i \geq n_2$. We have an absurdity, so $a_1 + a_2 \in J$. Hence, $J$ is an ideal, and $J = R \setminus R^\times$. So $J$ is the unique maximal ideal, and $R$ is local.

Exercise 3.3.4. Let $A$ be a finite dimensional $K$-algebra. Then $A$ is a local if and only if $A/J(A)$ is a division algebra.
Chapter 4

Indecomposable modules

4.1 The Krull–Remak–Schmidt decomposition theorem

Definition 4.1.1 ((In)decomposable module). Let $A$ be a finite dimensional $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field. A left $A$-module $M$ is decomposable if $M = M_1 \oplus M_2$, for $M_1, M_2 \subseteq M$ non zero. It is indecomposable otherwise, if it is not null.

Example 4.1.2. Any simple module is indecomposable.

Exercise 4.1.3. If conversely every indecomposable $A$-module is simple, then $A$ is a semisimple algebra.

 Remark 4.1.4. Every finitely generated $A$-module can be decomposed as a direct sum $M = \bigoplus_i M_i$ of indecomposable submodules.

Lemma 4.1.5 (Fitting’s lemma). Let $A$ be a finite dimensional $\mathbb{K}$-algebra, and $M$ a finitely generated $A$-module. Let $\phi \in \text{end}_A M$. Then there exists $n \in \mathbb{N}$ such that $M = \ker \phi^n \oplus \text{im} \phi^n$.

Proof. Since $\dim_{\mathbb{K}} M$ is finite, the following two series must stop:

$$M \supseteq \text{im} \phi \supseteq \text{im} \phi^2 \supseteq \cdots \supseteq \text{im} \phi^n = \text{im} \phi^{n+1} = \cdots,$$

$$0 \subseteq \ker \phi \subseteq \ker \phi^2 \subseteq \cdots \subseteq \ker \phi^n = \ker \phi^{n+1} = \cdots,$$

and $\text{im} \phi^{n+k} = \text{im} \phi^n$, $\ker \phi^{n+k} = \ker \phi^n$ for all $k \in \mathbb{N}$. Let $x \in M$. Then $\phi^n(x) = \phi^{2n}(y)$ for some $y \in M$ (as $\text{im} \phi^n = \text{im} \phi^{2n}$). So $\phi^n(x - \phi^n(y)) = 0$, therefore $x = \phi^n(y) + x - \phi^n(y)$. So $M = \text{im} \phi^n + \ker \phi^n$. Let $x \in \text{im} \phi^n \cap \ker \phi^n$. Then $x = \phi^n(z)$ for some $z \in M$. Therefore $0 = \phi^n(x) = \phi^{2n}(z)$. So $z \in \ker \phi^{2n} = \ker \phi^n$, hence $x = 0$. Hence $\text{im} \phi^n \cap \ker \phi^n = 0$, and we get $M = \ker \phi^n \oplus \text{im} \phi^n$. \qed
Theorem 4.1.6. Let $A$ be a finite dimensional $\mathbb{K}$-algebra, and $M$ a finitely generated left $A$-module. Then $M$ is indecomposable if and only if $\text{end}_A M$ is local.

Proof. $\implies$ Suppose $M$ indecomposable. By Fitting’s lemma, there exists an integer such that $M = \text{im} \phi^n \oplus \ker \phi^n$, for a given $\phi \in \text{end}_A M$. However, $M$ is indecomposable, so one of the summands is zero. If $\ker \phi^n = 0$ and $\text{im} \phi^n = M$, then $\ker \phi = 0$, and $\text{im} \phi = M$, and so $\phi$ is an automorphism. If $\ker \phi^n = M$, then $\phi$ is nilpotent. By lemma 3.3.3, we obtain that $\text{end}_A M$ is local.

$\Leftarrow$ Suppose $\text{end}_A M$ local. Let $M = M_1 \oplus M_2$, and $\pi_1 : M \longrightarrow M_1 \hookrightarrow M$ be the projections. Then $\pi_1^2 = \pi_1$, $\pi_1 + \pi_2 = \text{id}_M$. Without loss of generality, suppose $M_1 \neq 0$, then $\pi_1 \neq 0$, and so it is not nilpotent (because it is idempotent). Hence, $\pi_1$ is invertible. Then

$$\pi_1 = \pi_1^{-1} \pi_1^2 = \text{id}_M.$$ 

Hence $\pi_2 = 0$, and so $M_2 = 0$. \hfill $\square$

Corollary 4.1.7. $A$ is local if and only if $A A$ is indecomposable.

Proof. We have $\text{end}_A(AA) \cong A^{\text{op}}$ which is also local. \hfill $\square$

Theorem 4.1.8 (Krull–Remak–Schmidt). Let $A$ be a finitely generated $\mathbb{K}$-algebra, and $M$ a finite dimensional left $A$-module. Suppose that $M = \bigoplus_{i=1}^r M_i = \bigoplus_{j=1}^s N_j$, where all summands are indecomposable. Then $r = s$, and there exists $\sigma \in \mathfrak{S}_r$ such that $M_i \cong N_{\sigma(i)}$. In other words, a decomposition into indecomposable modules is essentially unique.

Proof. We proceed by induction on $r$. If $r = 1$, then $M = M_1$ is indecomposable, so $s = 1$, and $M_1 = N_1$. Suppose $r \geq 2$. Let $\varepsilon_i : M_1 \hookrightarrow M$ the canonical inclusion, $\pi_i : M \longrightarrow M_i$ the canonical projection, and $\eta_j : N_j \longrightarrow M$, $\rho_j : M \longrightarrow N_j$ the analogous. Then $\varepsilon_i \pi_i = \text{id}_{M_i}$, and $\varepsilon_i \pi_i : M \longrightarrow M$ is idempotent. Moreover $\sum_i \varepsilon_i \pi_i = \text{id}_M$. Similarly for $\eta_j$ and $\rho_j$.

The composite $M_1 \xleftarrow{\varepsilon_1} M \xrightarrow{\sum_j \eta_j \rho_j} M \xrightarrow{\pi_1} M$ is $\text{id}_{M_1}$. Therefore $\text{id}_{M_1} = \sum_j \pi_1 \eta_j \rho_j \varepsilon_1 \in \text{end}_A M_1$, and the latter ring is local. So it can’t be that all summands are nilpotent. Let $1 \leq j \leq s$ such that $\phi = \pi_1 \eta_j \rho_j \varepsilon_1$ is invertible. Without loss of generality (i.e. up to permutation), $j = 1$.

Let $\alpha = \rho_1 \varepsilon_1 \phi^{-1}$ and $\beta = \pi_1 \eta_1$. The composite $M_1 \xrightarrow{\alpha} N_1 \xrightarrow{\beta} M_1$ is $\text{id}_{M_1}$. Moreover $\alpha \beta$ is idempotent and not zero (because $\beta(\alpha \beta) \alpha = \text{id}_{M_1}$). It is therefore not nilpotent, so it is invertible, as $\text{end}_A N_1$ is local. Denote $\gamma = \alpha \beta$. Then $\gamma = \gamma^{-1} \gamma^2 = \gamma^{-1} \gamma = \text{id}_{N_1}$. So $\alpha$ and $\beta$ are mutually inverse, hence $M_1 \cong N_1$.

Now, $\bigoplus_{i=2}^r M_i \cong \bigoplus_{j=2}^s N_i$, and we apply the induction hypothesis. \hfill $\square$
4.2. IDEMPOTENT ELEMENTS

Remarks 4.1.9. 1. The Krull–Remak–Schmidt theorem fails for some other rings, e.g. for ring of integers in a number field (a finite field extension of $\mathbb{Q}$), e.g. $\mathbb{Z}[\sqrt{5}]$.

2. But it works for PIDs (yayyy).

3. If $A$ is a finitely generated $\mathbb{K}$-algebra, and $M$ is a finite dimensional left $A$-module, then we know that $\dim_{\mathbb{K}} M < \infty$. Therefore a decomposition into indecomposable always exists (by induction).

4.2. Idempotent elements

Definition 4.2.1 (Orthogonal/primitive idempotents). Let $R$ be a ring, and $e, f \in R$ two idempotents. They are called orthogonal if $ef = fe = 0$. An idempotent $e \in R$ is called primitive if $e$ cannot be decomposed as a sum of two nonzero orthogonal idempotents.

Example 4.2.2. In any ring $R$, and for all idempotent $e \in R$, we have that $e$ and $(1 - e)$ are orthogonal idempotents. Hence, if $e \neq 0, 1$, $e + (1 - e) = 1$ is an orthogonal decomposition, and 1 is not primitive.

Lemma 4.2.3. Let $R$ be a ring.

1. If $R = \bigoplus_{i=1}^{r} Q_i$, then $Q_i = Re_i$, where $e_i$ is idempotent, and $\sum_i e_i = 1$ is an orthogonal decomposition.

2. Conversely, any orthogonal decomposition of 1 into idempotents $e_1, \ldots, e_r$ leads to a decomposition $R = \bigoplus_{i=1}^{r} Re_i$.

3. If $e \in R$ is idempotent, then $Re$ is indecomposable if and only if $e$ is primitive.

Proof. 1. There is a unique way of writing $1 = \sum_i e_i$, where $e_i \in Q_i$. Then $e_j = \sum_i e_j e_i$, and so $e_j e_i = 0$ if $i \neq j$, and $e_j^2 = e_j$. So $\{e_i\}_{i \in Q_i}$ is a family of orthogonal idempotents. Take $x \in Q_j$. Then $x = x1 = \sum_i xe_i$, and so $xe_i = 0$ if $i \neq j$, and $xe_j = x$. So $Q_j = Re_j$.

2. We have $1 = \sum_i e_i$, so $R = R1 = \sum_i Re_i$. Let $x \in Re_j \cap \sum_{i \neq j} Re_i$. Then $x = xe_i$, and $x = \sum_{i \neq j} j_i e_j$. So $x = xe_i = \sum_{i \neq j} j_i e_i e_i = 0$.

3. If $e = u + v$ is a decomposition into orthogonal idempotents, then $Re = Ru \oplus Rv$. Conversely, if $Re = U \oplus V$, then $e = \pi_U(e) + \pi_V(e)$, and by the same argument as in point 1., those two terms are orthogonal idempotents.
Corollary 4.2.4. Let $A$ be a finite dimensional $\mathbb{K}$-algebra. Let $P$ be a finitely generated projective left $A$-module.

1. $P$ is indecomposable if and only if $P \cong Ae$, where $e$ is a primitive idempotent of $A$.

2. There are finitely many indecomposable projective modules up to isomorphism.

Proof. First, decompose $A$. By the previous lemma, $A = \bigoplus_{i=1}^{n} Ae_i$, where $1 = \sum_i e_i$ is an orthogonal primitive idempotent decomposition. Such a decomposition must exist because of the Krull–Remak–Schmidt theorem.

1. $P$ is a direct summand of $A^r$ which decomposes as $\bigoplus_{i=1}^{n} (Ae_i)^r$. By the Krull–Remak–Schmidt theorem, $P \cong Ae_i$ for some $i$.

2. We know that $A^r \cong \bigoplus_{i=1}^{n} (Ae_i)^r$, and so any indecomposable projective module is isomorphic to $Ae_i$, for some $e_i$.

\[ \square \]

Theorem 4.2.5. Let $A$ be a finite dimensional $\mathbb{K}$-algebra. There is a bijection

\[
\left\{ \text{conjugacy classes of primitive idempotents in } A \right\} \longleftrightarrow \left\{ \text{iso. classes of indecomposable projective left } A\text{-modules} \right\}
\]

mapping the class of a primitive idempotent $e \in A$ to the isomorphism class of $Ae$.

Proof. We have to check that this mapping is well defined. It is clear that any conjugate of an idempotent (resp. primitive idempotent) is again idempotent (resp. primitive idempotent). Let $e, f \in A$ be conjugate primitive idempotents. There exists $u \in A^\times$ such that $e = uf u^{-1}$. Define $m_u : Af \rightarrow Ae : x \mapsto xu$, and $m_{u^{-1}} : Ae \rightarrow Af$ similarly. Then $m_u$ and $m_{u^{-1}}$ are mutually inverse, and so $Ae \cong Af$.

Then, the mapping is surjective by the previous corollary. It remains to show injectivity. Let $e, f \in A$ be primitive idempotents such that $Ae \cong Af$. We have $A \cong Ae \oplus A(1 - e) \cong Af \oplus A(1 - f)$. So $A(1 - e) \cong A(1 - f)$ by cancellation (exercise 2, sheet 8). Therefore there exists an isomorphism $\phi : A \rightarrow A$ such that $\phi(Ae) = Af$, and $\phi(A(1 - e)) = A(1 - f)$. However, $\phi$ must be of the form $\phi = m_u$ (right multiplication by $u$), for some $u \in A^\times$. Then $u^{-1}eu$ is an idempotent, and $Au^{-1}eu = Aeu = \phi(Ae) = Af$, and $A(1 - u^{-1}eu) = Au^{-1}(1 - e)u = \phi(A(1 - e)) = A(1 - f)$. Then $m_f$ and $m_{u^{-1}}eu$ are the identity on $Af$, and zero on $A(1 - f)$. Therefore $m_f = m_{u^{-1}}eu$ on $A = Af \oplus A(1 - f)$. In particular, they coincide on $1$, and so $f = u^{-1}eu$.

\[ \square \]
Example 4.2.6. 1. Take $A$ to be semisimple. By the Wedderburn theorem, $A \cong \prod_i M_{n_i}(D_i)$, where $D_i$ is a division ring. Take

$$e_i = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{n_i}(D_i).$$

It is a primitive idempotent. Moreover, as $A$ is semisimple, any indecomposable projective is simple. Hence

$$Ae_i \cong M_{n_i}(D_i)e_i = \begin{pmatrix} D_i \\ \vdots \\ D_i \end{pmatrix} \cong D_i^n$$
is simple. See exercises.

2. An example with infinitely many indecomposable modules. Define the algebra $A = \mathbb{K}[X,Y]/(X^2,Y^2,XY)$. Let $s = X$, $t = Y$. Then $\dim_\mathbb{K} A = 3$, with basis $\{1,s,t\}$, and multiplication $s^2 = t^2 = st = ts = 0$. Clearly, $J(A) = \mathbb{K}s \oplus \mathbb{K}t$, as it is the largest nilpotent ideal. Then, $A/J(A) = \mathbb{K}$. We have only one simple module $S = A/J(A)$, one idempotent, namely $1$, which is also primitive. Therefore, $A$ is indecomposable projective. Fix $\lambda \in \mathbb{K}^\times$, and define a 2-dimensional $A$-module $M_\lambda$ with basis $\{m,n\}$ as follows:

$$sm = n, \quad sn = 0$$

$$tm = \lambda n, \quad tn = 0.$$ 

Let $N_\lambda = J(M_\lambda) = J(A)M_\lambda = \mathbb{K}n$. We have $N_\lambda \cong S$, and $M_\lambda/N_\lambda \cong S$, and so there is a short exact sequence

$$0 \to S \to M_\lambda \to S \to 0.$$ 

The rest of the proof that $M_\lambda$ is indecomposable is left as an exercise. Next, $m_s : M_\lambda \to M_\lambda : x \mapsto sx$ induces a linear map $\tilde{m}_s : M_\lambda/N_\lambda \to N_\lambda : sx \mapsto x$. Moreover $\tilde{m}_s(m) = n$, and so $\tilde{m}_s$ is an isomorphism. Similarly, $m_t : M_\lambda \to M_\lambda$ induces a $\mathbb{K}$-linear isomorphism $\tilde{m}_t : M_\lambda/N_\lambda \to N_\lambda$, with $\tilde{m}_t(m) = \lambda n$. Now the composite $\phi_\lambda$

$$N_\lambda = J(M_\lambda) \xrightarrow{\tilde{m}_t^{-1}} M_\lambda/J(M_\lambda) \xrightarrow{\tilde{m}_s} N_\lambda$$

maps $n$ to $\lambda^{-1}n$. The map $\phi_\lambda$ is intrisically defined by $M_\lambda$, because it is $\tilde{m}_s\tilde{m}_t^{-1}$. Associated to $M_\lambda$ we have a intrisically defined map $\phi_\lambda : N_\lambda = \mathbb{K}n \to N_\lambda$ which is multiplication by $\lambda^{-1}$. So from $M_\lambda$, one can recover $\lambda$. Therefore if $M_\lambda \cong M_\mu$, then $\lambda = \mu$. Finally, if $\mathbb{K}$ is infinite, we have infinitely many non isomorphic indecomposable modules.
Chapter 5

Lifting idempotents

Definition 5.0.1 (Primitive orthogonal decomposition). Let $R$ be a ring. A primitive orthogonal decomposition of an idempotent $e \in R$ is a decomposition of $e$ as a sum of primitive pairwise orthogonal idempotents.

Lemma 5.0.2. Let $A$ be a finite dimensional $\mathbb{K}$-algebra, $e \in A$ idempotent. Consider the $\mathbb{K}$-algebra $eAe$ (with identity $e$). Then $J(eAe) = J(A)\cap eAe$.

Proof. First, $J(A)e \subseteq J(A), eAe$. Next, if $a \in J(A) \cap eAe$, then $a = ebe$ with $b \in A$, and so $a = eae$. Therefore $a \in eJ(A)e$. Hence, $eJ(A)e = J(A) \cap eAe$.

Moreover, $J(A)^N = 0$ for some large enough $N \in \mathbb{N}$, so $(eJ(A)e)^N = 0$. So the two sided ideal $eJ(A)e$ of $eAe$ is nilpotent, hence $eJ(A)e \subseteq J(eAe)$. Next, $AJ(eA)eA$ is the two sided ideal of $A$ generated by $J(eAe)$. We have

$$(AJ(eA)eA)^2 = A(eJ(eA)eA)(eJ(eA)eA)eA = J(eAe).$$

Similarly, $(AJ(eA)eA)^n = A(eJ(eA)eA)^n eA$, hence, $AJ(eA)eA$ is nilpotent, and so $AJ(eA)eA \subseteq J(A)$. Therefore, $J(eAe) \subseteq J(A)$, hence $J(eAe) \subseteq eJ(A)e$.

Finally, $J(eAe) = eJ(A)e$. \hfill \square

Theorem 5.0.3 (Lifting stuffs). Let $A$ be a finite dimensional $\mathbb{K}$-algebra, $\bar{A} = A/J(A)$, and write $\bar{a} \in \bar{A}$ for the class of $a \in A$ in $\bar{A}$.

1. Lifting invertibility: $a \in A$ is invertible if and only if $\bar{a}$ is invertible. In other words, there is a short exact sequence of groups

$$\{1\} \rightarrow 1 + J(A) \rightarrow A^\times \rightarrow \bar{A}^\times \rightarrow \{1\}.$$

2. Lifting idempotents: For any idempotent $g \in \bar{A}$, there exists an idempotent $e \in A$ such that $\bar{e} = g$. 

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3. **Lifting conjugacy of idempotents** : Let \( e, f \in A \) be two idempotents, if \( \bar{e} \) and \( \bar{f} \) are conjugate in \( \bar{A} \), then \( e \) and \( f \) are conjugate in \( A \). More precisely, if \( \bar{f} = \bar{u}\bar{e}\bar{u}^{-1} \), then \( \bar{u} \) can be lifted as \( u \in \bar{A}^\times \) such that \( f = ueu^{-1} \). In particular, if \( \bar{e} = \bar{f} \), then there exists \( u \in 1 + J(A) \) such that \( f = ueu^{-1} \).

4. **Lifting primitivity** : Let \( e \in A \) be idempotent. Then \( e \) is primitive in \( A \) if and only if \( \bar{e} \) is primitive in \( \bar{A} \).

5. **Lifting idempotent decompositions** : Let \( e \in A \) be idempotent, and \( \bar{e} = \sum_i g_i \) be a primitive orthogonal decomposition of \( \bar{E} \) in \( \bar{A} \). Then there exists \( e_i \in A \) with \( \bar{e}_i = g_i \), and such that \( e = \sum_i e_i \) is a primitive orthogonal decomposition of \( e \) in \( A \).

**Proof.** 1. If \( a \in A \) is invertible, then clearly \( \bar{a} \) is also invertible. Conversely, take \( a \in A \) such that \( \bar{a} \) is invertible, and suppose that \( a \) is not invertible. If \( a \) is not in a maximal left ideal, then \( Aa = A \), and similarly on the right. If \( Aa = aA = A \), then \( \exists b, c \in A \) such that \( ba = ac = 1 \), so \( a \) is invertible, a contradiction. Therefore, \( a \) belongs to a maximal left ideal \( m \subseteq A \). By definition, \( J(A) \subseteq m \). Hence \( \bar{m} = m/J(A) \) is a maximal ideal in \( \bar{A} \), and \( \bar{a} \in \bar{m} \). Consequently \( \bar{a} \) is not invertible, a contradiction.

Finally, the projection \( A^\times \to \bar{A}^\times \) is surjective, with kernel \( 1 + J(A) \).

2. Let \( g \in \bar{A} \) be an idempotent. By surjectivity of the projection \( A \to \bar{A} \), there exists \( a_1 \in A \) such that \( \bar{a}_1 = g \). Let \( b_1 = a_1^2 - a_1 \). Define inductively \( a_n = a_{n-1} + b_{n-1} - 2a_{n-1}b_{n-1} \), and \( b_n = a_n - a_n^2 \).

We show by induction that \( b_n \in J(A)^n \). For \( n = 1 \), remark that \( \bar{b}_1 = g^2 - g = 0 \), and so \( b_1 \in J(A) \). We use the fact that \( a_n^2 = a_n + b_n \), and also that \( n \) and \( a_n \) and \( b_n \) commute (as \( b_n = a_n^2 - a_n \)). By induction, assume that \( b_{n-1} \in J(A)^{n-1} \). Notice that \( b_n^2 \in J(A)^{2n-1} \subset J(A)^{n+1} \). Compute \( a_{n+1} \) modulo \( J(A)^{n+1} \).

\[
a_{n+1}^2 = (a_n + b_n - 2a_nb_n)^2 = a_n^2 + b_n^2 + 4a_n^2b_n^2 + 2a_nb_n - 4a_n^2b_n - 4a_n^2b_n^2
\equiv a_n^2 + 2a_nb_n - 4a_n^2b_n \pmod{J(A)^{n+1}}
\equiv a_n + b_n + 2a_nb_n - 4(a_n + b_n)b_n \pmod{J(A)^{n+1}}
\equiv a_n + b_n - 2a_nb_n \pmod{J(A)^{n+1}}
\equiv a_{n+1} \pmod{J(A)^{n+1}}.
\]

Therefore, \( a_{n+1}^2 \equiv a_{n+1} \pmod{J(A)^{n+1}} \), and so \( b_{n+1} = a_{n+1}^2 - a_{n+1} \in J(A)^{n+1} \), thus proving the claim.

As \( J(A) \) is nilpotent, there exists, \( N \in \mathbb{N} \) such that \( J(A)^N = 0 \), and so \( b_N = a_N^2 - a_N = 0 \), and \( a_N \) is idempotent. By induction, on can easily prove that \( a_N = g \), which proves the statement.
3. Let $e, f \in A$ be two idempotents such that $\tilde{f} = \tilde{u} \tilde{e} \tilde{u}^{-1}$ for some $\tilde{u} \in \tilde{a}^\times$. Then $\tilde{u}$ lifts as $u \in A^\times$ by part 1. Denote by $h = u e u^{-1}$. It is an idempotent, and $\tilde{h} = \tilde{f}$. Let $v = 1 - h - f + 2h f$. Then $\tilde{v} = 1 - \tilde{h} - f + 2f \tilde{h} = 1$. So $v \in 1 + J(A)$ is invertible. Compute

$$
\begin{align*}
hev &= h - h - hf + 2hf = hf \\
 vf &= f - hf - f + 2hf = hf.
\end{align*}
$$

So $hv = vf$, and $ueu^{-1} = h = vfv^{-1}$, and $e$ and $f$ are conjugate.

4. Let $e \in A$ to be an idempotent. If $e$ is not primitive, then $e = f_1 + f_2$, for $f_1, f_2 \in A$ orthogonal idempotents. Remark that $f_1, f_2 \notin J(A)$ as $J(A)$ is nilpotent. So $\tilde{f}_1, \tilde{f}_2 \neq 0$, and as $\tilde{e} = \tilde{f}_1 + \tilde{f}_2$ is an orthogonal idempotent decomposition, we have that $\tilde{e}$ is not primitive.

Conversely, if $\tilde{e}$ is not primitive, then $\tilde{e} = \tilde{f}_1 + \tilde{f}_2$, for $\tilde{f}_1, \tilde{f}_2 \in \tilde{a}$ orthogonal idempotents. Consider the $K$-algebra $eAe$. By previous lemma, we have $eAe = eAe$. Notice that $f_1, f_2 \in eAe$. By part 2, $\tilde{f}_1$ can be lifted as an idempotent $f_1 \in eAe$. Define $f_2 = e - f_1$, which lifts $f_2$. Then $e = f_1 + f_2$ is an orthogonal decomposition of $e$, and it is not primitive.

5. Let $e \in A$ be an idempotent, and $\tilde{e} = \sum_{i=1}^{r} g_i$ be an orthogonal decomposition. We prove the statement by induction on $r$. If $r = 1$ there is nothing to prove. We work in the $K$-algebra $eAe$. We have that $g_1 \in eae$ as before, and we can lift it as an idempotent $f_1 \in eAe$. Then $e = f_1 + (e - f_1)$ is an orthogonal decomposition (in $eAe$).

Hence $\tilde{e} = g_1 + \sum_{i=2}^{r} g_i$. By induction hypothesis, the decomposition

$$
\tilde{e} - f_1 = \sum_{i=2}^{r} g_i \text{ lifts as an orthogonal decomposition } e - f_1 = \sum_{i=2}^{r} f_i.
$$

So $e = \sum_{i=1}^{r} f_i$ is an orthogonal decomposition lifting $\tilde{e} = \sum_{i=1}^{r} g_i$.

\[\square\]

**Theorem 5.0.4.** Let $A$ be a finite dimensional $K$-algebra.

1. If $e \in A$ is a primitive idempotent, then the indecomposable projective $A$-module $Ae$ has a unique maximal submodule $J(A)e$. In other words, $Ae$ has a unique simple quotient, namely $Ae / J(A)e = \bar{Ae}$.

2. Let $e, f \in A$ be two primitive idempotents, then $Ae \cong Af$ if and only if $Ae \cong \bar{Af}$.

3. There is a bijection

$$
\begin{align*}
\{ & \text{iso. classes of} \ \text{indec. projective } A\text{-mods.} \} \leftrightarrow \{ & \text{iso. classes of} \ \text{simple } A\text{-mods.} \}.
\end{align*}
$$
mapping the class of $Ae$ to the class of $\bar{A}\bar{e}$.

Proof. The $\mathbb{K}$-algebra projection map $A \rightarrow \bar{A}$ sends $Ae$ to $\bar{A}\bar{e}$. If $e$ is primitive, then so is $\bar{e}$, by previous theorem. So $\bar{A}\bar{e}$ is a projective indecomposable $\bar{A}$-module. However, $\bar{A}$ is semisimple. So $\bar{A}\bar{e}$ is simple as a $\bar{A}$-module, and also as an $A$-module (with $J(A)$ acting as zero, the classical restriction of scalars). Moreover, any simple $A$-module has this form, because $J(A)$ must act by zero on it (by definition of the Jacobson radical).

1. We have seen that $J(A)e$ is a maximal submodule of $Ae$, because $\bar{A}\bar{e}$ is simple. Let $M$ be a maximal submodule of $Ae$. Then $Ae/M$ is simple, therefore $J(A)(Ae/M) = 0$, and so $J(A)Ae \leq M$, that is $J(A)e \leq M \leq Ae$. Since $J(A)e$ is maximal, we have $M = J(A)e$.

2. We use the bijection of theorem 4.2.5. We have $Ae \cong Af$ if and only if $e$ and $f$ are conjugate in $A$, if and only if $\bar{e}$ and $\bar{f}$ are conjugate in $\bar{A}$, if and only if $\bar{A}\bar{e} \cong \bar{A}\bar{f}$.

3. This is a consequence of the previous points.
Chapter 6

The Wedderburn–Malcev theorem

Definition 6.0.1 (Split algebra). Let \( A \) be a finite dimensional \( \mathbb{K} \)-algebra. It is said to be split if

1. it is semisimple,
2. \( \text{end}_A S = \mathbb{K} \), for every simple \( A \)-module \( S \).

Remark 6.0.2. Take \( A \) a semisimple \( \mathbb{K} \)-algebra. The by Wedderburn theorem, we have that

\[ A \cong \prod_i M_{n_i}(D_i). \]

Then \( A \) is split if and only if \( D_i = \mathbb{K} \), since \( D_i = (\text{end}_A S_i)^{\text{op}} \).

Examples 6.0.3. 1. If \( \mathbb{K} \) is algebraically closed, then every semisimple algebra is split, by Schur’s lemma.

2. Consider \( \mathbb{R}Q_8 \), the \( \mathbb{R} \)-group algebra of the quaternion group of order 8. Then \( \mathbb{R}Q_8 \cong \mathbb{R}^4 \times \mathbb{H} \), which is non split (see exercise 2 of sheet 5).

Lemma 6.0.4. Let \( A = A_1 \times A_2 \) be a direct product of two finite dimensional \( \mathbb{K} \)-algebras. Let \( e \in A \) be a primitive idempotent.

1. Then either \( e = (e_1, 0) \), where \( e_1 \in A_1 \) is a primitive idempotent, or \( e = (0, e_2) \), where \( e_2 \in A_2 \) is a primitive idempotent.

2. A primitive orthogonal decomposition of \( 1 \in A \) is obtained by a primitive orthogonal decomposition of \( 1 \in A_1 \) and \( 1 \in A_2 \).

Proof. 1. Let \( e \in (e_1, e_2) \in A \) be a primitive idempotent. Then \( e^2 = e \), so \( e_1 \) and \( e_2 \) are idempotents, and so are \( (e_1, 0) \) and \( (0, e_2) \). Moreover, \( e = (e_1, 0) + (0, e_2) \) is an orthogonal decomposition, so either \( e_1 = 0 \), or \( e_2 = 0 \).
2. Clear since $1_A = (1_{A_1}, 1_{A_2}) = (1_{A_1}, 0) + (0, 1_{A_2})$ is an (not necessarily primitive) orthogonal decomposition.

Lemma 6.0.5. Let $A = M_n(K)$ be a split simple algebra over $K$.

1. An idempotent $e \in A$ is primitive if and only if $e$ is a projection matrix onto a one dimensional subspace of $K^n$.

2. Every primitive idempotents are conjugate.

3. The number of primitive idempotents in a primitive orthogonal decomposition of $I_n$ is $n$. More precisely, $I_n = \sum_{j=1}^{n} E_{j,j}$. By part 2, there exists $U_j \in \text{GL}_n(K)$ such that $E_{j,j} = U_j^{-1} E_{1,1} U_j$.

4. The elements $\{U_j^{-1} E_{1,1} U_k\}$ form a $K$-basis of $M_n(K)$. If the $U_j$ are transposition matrices, then the basis is precisely the canonical basis.

Proof. Already done in an exercise sheet. For point 4. : if $U_j$ are permutation matrices, then the result is clear. If not, then

$$(U_j^{-1} E_{1,1} U_k)(U_p^{-1} E_{1,1} U_q) = U_j^{-1} U_k (U_p^{-1} E_{1,1} U_k)(U_p^{-1} E_{1,1} U_p) U_p^{-1} U_q$$

$$= U_j^{-1} U_k E_{k,k} E_{p,p} U_p^{-1} U_q$$

is either 0 or another element of the set. We hence have an orthogonality relation. \qed

Theorem 6.0.6 (Wedderburn–Malcev). Let $A$ be a finite dimensional $K$-algebra such that $A/J(A)$ is a split algebra. Let $\pi : A \to A/J(A) : a \mapsto \bar{a}$ be the quotient map.

1. There is a semisimple subalgebra $S \subseteq A$ such that $\pi|_S$ is an isomorphism. In other words there is a section $\sigma : A/J(A) \to A$ of $\pi$.

2. If $T \subseteq A$ is another semisimple subalgebra such that $\pi|_T$ is an isomorphism, then $T$ and $S$ are conjugate. In other words, the section of $\pi$ is unique up to conjugacy.

Remark 6.0.7. Let $A$ be a finite dimensional $K$-algebra. Then $A$ is separable if

1. $A$ is semisimple,

2. consider the Wedderburn decomposition $A \cong \prod_i M_{n_i}(D_i)$, then $Z(D_i)/K$ is a separable extension.

The Wedderburn–Malcev theorem also holds for algebra $A$ such that $A/J(A)$ is separable.
Chapter 7

Symmetric algebras

7.1 Definition

Definition 7.1.1. Let $A$ be a finite dimensional $\mathbb{K}$-algebra. Let $M$ be a finitely generated left $A$-module. The dual of $M$ is defined as $M^* = \text{Hom}_\mathbb{K}(M, \mathbb{K})$ endowed with the following structure of right $A$-module: if $f \in M^*$ and $a \in A$, then

$$fa : M \longrightarrow \mathbb{K}$$

$$m \longmapsto f(am).$$

Proposition 7.1.2. Let $A$ be a finite dimensional $\mathbb{K}$-algebra. The following are equivalent:

1. There is an isomorphism of right $A$-modules $\phi : A_A \longrightarrow (A A)^*$ which is symmetric: $\phi(a)(b) = \phi(b)(a)$, for every $a, b \in A$.

2. There exists a symmetric non degenerate bilinear form $\beta$ on $A$ that is symmetric and associative, i.e. such that $\beta(ab,c) = \beta(a,bc)$, for every $a, b, c \in A$.

3. There is a linear form $\lambda : A \longrightarrow \mathbb{K}$ which is symmetric, i.e. $\lambda(ab) = \lambda(ba)$, $\forall a, b \in A$, such that $\ker \lambda$ doesn’t contain any right ideal of $A$.

Proof.

1. $\iff$ 2. From a symmetric $\phi : A_A \longrightarrow (A A)^*$ we can construct a symmetric $\beta$ defined as $\beta(a,b) = \phi(a)(b)$. Then, $\phi$ is an isomorphism if and only if $\beta$ is non degenerate (as we are in finite dimensional vector spaces). The associativity is routine verifications.

2. $\iff$ 3. If $\beta : A \times A \longrightarrow \mathbb{K}$ is symmetric associative, then define $\lambda$ by $\lambda(a) = \beta(a,1)$. Since $\beta$ is associative, we have $\lambda(ab) = \beta(ab,1) = \beta(a,b)$. 

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Since $\beta$ is symmetric, $\lambda$ is too. Conversely, if $\lambda$ is linear symmetric, then define $\beta(a, b) = \lambda(ab)$. Since $\lambda$ is symmetric, $\beta$ is too. Moreover, $\beta$ is associative because multiplication in $A$ is too. Consider the following:

\[
\begin{align*}
\beta(a, x) &= 0 \quad \forall x \in A \\
\iff \quad \lambda(ax) &= 0 \quad \forall x \in A \\
\iff \quad aA & \subseteq \ker \lambda \\
\iff \quad \ker \lambda & \text{ cont. a right ideal cont. } a.
\end{align*}
\]

Then it is clear that $\beta$ is non degenerate if and only if $\ker \lambda$ doesn’t contain any right ideal.

\[\square\]

**Definition 7.1.3** (Symmetric algebra). Let $A$ be a finite dimensional $K$-algebra. Then $A$ is called symmetric if one (and hence all) of the conditions of proposition 7.1.2 hold. If this case, the linear form $\lambda$ is called a symmetrizing form\(^1\). A symmetrizing form may not be unique.

**Examples 7.1.4.**

1. $A = M_n(\mathbb{K})$ is symmetric with symmetrizing form $\text{tr} : M_n(\mathbb{K}) \rightarrow \mathbb{K}$.

   Indeed, we know that $\text{tr}$ is linear and that $\text{tr}(XY) = \text{tr}(YX)$, $\forall X, Y \in M_n(\mathbb{K})$. Next, we have a canonical basis $\{E_{p,q}\}_{p,q}$ of $M_n(\mathbb{K})$. In order to prove that the associated bilinear form $\beta(X, Y) = \text{tr}(X, Y)$ is non degenerate, it suffices to find a dual basis. Here it is $\{E_{p,q}\}_{p,q}$ because $E_{p,q}E_{r,s} = \delta_{q,r}E_{p,s}$, and therefore

   \[
   \beta(E_{p,q}E_{r,s}) = \text{tr}(E_{p,q}E_{r,s}) = \delta_{q,r}\delta_{p,s}.
   \]

   Then clearly, $\beta$ is non degenerate. Remark that a typical right ideal in $M_n(\mathbb{K})$ has form

   \[
   \begin{bmatrix}
   0 \\
   \vdots \\
   \vdots \\
   0
   \end{bmatrix}
   \notin \ker \text{tr}.
   \]

2. Let $G$ be a finite group. The group algebra $\mathbb{K}G$ is symmetric with symmetrizing form $\lambda : \mathbb{K}G \rightarrow \mathbb{K}$

   \[
   g \mapsto \begin{cases} 
   1 & \text{if } g = 1 \\
   0 & \text{ow}.
   \end{cases}
   \]

---

\(^1\)“forme symétrisante” in french
Then \( \lambda(gh) = 1 \) if and only if \( g \) and \( h \) are mutually inverse. Hence \( \lambda(hg) = 1 \) if and only if \( g \) and \( h \) are mutually inverse. By linearity, \( \lambda \) is symmetric. The corresponding bilinear form is given by

\[
\beta : \mathbb{K}G \times \mathbb{K}G \to \mathbb{K} \\
(g, h) \mapsto \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{ow.} \end{cases}
\]

Remark that \( \{g^{-1}\}_{g \in G} \) is a dual basis of \( \{g\}_{g \in G} \), and hence \( \beta \) is non degenerate.

### 7.2 Injective modules

**Lemma 7.2.1.** Let \( R \) be a ring and let \( I \) be a finitely generated left \( R \)-module. The following are equivalent:

1. For any injective homomorphism \( j : L \to M \) between finitely generated \( R \)-modules, and for any homomorphism \( \phi : L \to I \), there exists lift \( \tilde{\phi} : M \to I \) such that the following diagram commutes:

\[
\begin{array}{cccc}
L & \xrightarrow{j} & M \\
\phi \downarrow & & \tilde{\phi} \\
I & \xleftarrow{\phi^*} & I^*.
\end{array}
\]

2. Any injective homomorphism \( I \to M \) admit a retraction.

**Proof.**

1. \( \implies \) 2. Take \( L = I \) and \( \phi = \text{id}_I \).

2. \( \implies \) 1. With a huge loss of generality, we consider \( R \) to be a finite dimensional \( \mathbb{K} \)-algebra. Take \( L, M, j \) and \( \phi \) as in point 1. and apply duality:

\[
\begin{array}{cccc}
L^* & \xleftarrow{j^*} & M^* \\
\phi^* \downarrow & & \sim & \tilde{\phi}^* \\
I^*.
\end{array}
\]

Then \( I^* \) is projective, \( j^* \) is surjective, and there exists \( \tilde{\phi}^* \) lifting \( \phi^* \). Since we consider finite dimensional modules over \( \mathbb{K} \), dualization \((-)^* \) is involutive. Hence \( \tilde{\phi}^{**} \) lifts \( (\phi^*)^* = \phi \).

\( \square \)
**Definition 7.2.2** (Injective module). A $R$-module $I$ satisfying one (and hence all) condition of lemma 7.2.1 is called an injective module.

**Lemma 7.2.3.** Let $M$ be a finitely generated left $A$-module. Then $M$ is projective if and only if $M^*$ is injective as a right $A$-module.

**Proof.** Easy. 

**Exercise 7.2.4.** If $A$ is a symmetric algebra, the isomorphism $A A \cong (A A)^*$ of left $A$-modules is also an isomorphism of right modules.

**Proposition 7.2.5.** Let $A$ be a finite dimensional symmetric $K$-algebra. Then projective and injective modules coincide. We also say that $A$ is self-injective.

**Proof.** $A A$ is a free right $A$-module hence projective. Therefore $(A A)^*$ is an injective left $A$-module. Since $(A A)^* \cong A A$ we have that $A A$ is injective. Then $(A A)^{\oplus n}$ is injective (exercise), and any direct summand $P$ of $(A A)^{\oplus n}$ is injective (exercise again). Hence all projective modules are injective. Dualize for the converse.

**Definition 7.2.6** (Socle of a module). Let $M$ be a finitely generated $A$-module. Then the socle of $M$, written $soc M$ is the sum of all simple submodules of $M$. It is also the largest semisimple submodule of $M$.

**Remark 7.2.7.** 1. We know that indecomposable projective $A$-modules have a unique simple quotient. By duality, and using exercise 1 of sheet 12, we have that any indecomposable injective submodule have a unique simple submodule. Moreover, we have a bijection

$$\begin{array}{ccl}
\{ \text{iso. classes of indec. injective } A\text{-mods.} \} & \leftrightarrow & \{ \text{iso. classes of simple } A\text{-mods.} \}
\end{array}$$

which associates to an injective module $I$ its socle $soc I$.

2. If $A$ is symmetric, then projective and injective modules coincide, and therefore any projective indecomposable module $P$ have both a unique simple quotient $P/J(P)$ and a unique simple submodule $soc P$.

**Theorem 7.2.8.** Let $A$ be a symmetric finite dimensional $K$-algebra, and let $P$ be an indecomposable projective $A$-module. Then $P/J(P) \cong soc P$.

**Proof.** We know that $P \cong Ae$, where $e$ is a primitive idempotent of $A$, and that $P/J(P) = Ae/J(A)e$. The socle $soc Ae$ is a left ideal of $A$, hence not contained in $ker \lambda$, where $\lambda : A \to K$ is a symmetrizing form. Let $a \in soc Ae$ be such that $\lambda(a) \neq 0$. We have $a = ae$, therefore $0 \neq \lambda(a) = \lambda(ae) = \lambda(ea)$, and $ea \neq 0$.

$$\phi : Ae \to soc Ae$$

$$xe \mapsto xea.$$
Clearly, $\phi$ is a homomorphism of left $A$-modules. It is non zero as $\phi(e) = ea \neq 0$. As $\text{soc } Ae$ is simple, we have that $\phi$ is surjective. Moreover, $\phi$ induces an isomorphism $Ae / \ker \phi \cong \text{soc } Ae$. As $Ae$ have a unique simple quotient, we have that $Ae / J(A)e \cong Ae / \ker \phi$, and so $P / J(P) \cong \text{soc } P$. 

**Example 7.2.9.** Let $Q$ be a quiver without oriented cycles, so $\mathbb{K}Q$ is finite dimensional.

- **Trivial case:** no arrow. Then $\mathbb{K}Q \cong \mathbb{K}^n$, where $n$ is the number of vertex of $Q$. Hence $\mathbb{K}Q$ is split semisimple, hence symmetric (exercise 5.a, sheet 12).

- **Non trivial cases:** there is at least one arrow. We want to prove that $\mathbb{K}Q$ isn’t symmetric. Let $v$ be the target of an arrow, and $l_v$ be the empty path at $v$, which is a primitive idempotent of $\mathbb{K}Q$. Then $P_v = \mathbb{K}Q l_v$ is projective indecomposable (in fact, it is spanned by all paths ending at $v$). Also, $P_v / J(P_v) = \mathbb{K}Q l_v / J(\mathbb{K}Q) l_v$ is a one dimensional simple module generated by the class of $l_v$.

Let $u$ be the origin of a maximal path $\pi$ ending at $v$. Then there is no arrow with target $u$ be maximality. Therefore $\mathbb{K}Q l_u = \mathbb{K}l_u$ is simple. There is a homomorphism of left $\mathbb{K}Q$-modules

$$m_\pi : \mathbb{K}Q l_u = \mathbb{K}l_u \longrightarrow \mathbb{K}Q l_v$$

$$l_u \longmapsto l_u \pi.$$ 

Since $\mathbb{K}l_u$ is one dimensional and $l_u \pi \neq 0$, we have that $m_\pi$ is injective. Therefore the simple module $\mathbb{K}Q l_u = \mathbb{K}l_u$ corresponding to $l_u$ is isomorphic to a submodule of the projective module $\mathbb{K}Q l_v$ corresponding to $l_v$. Hence $\mathbb{K}Q l_v$ has a simple submodule $\mathbb{K}l_u$ in its socle, and this simple submodule is not isomorphic to $\mathbb{K}Q l_v / J(\mathbb{K}Q l_v)$. By the previous theorem, $\mathbb{K}Q$ is not symmetric.
Chapter 8

Finite representation type

8.1 Definition

Definition 8.1.1. Let $A$ be a finite dimensional $\mathbb{K}$-algebra. We say that $A$ has finite representation type if there are finitely many isomorphism classes of finitely generated indecomposable left $A$-modules.

Lemma 8.1.2. $A = \mathbb{K}[X]/(X^n)$ is symmetric, where $n \geq 1$. In particular, $AA$ is an injective module.

Proof. Let $x$ be the class of $X$ in $A$, so $x^n = 0$. Obviously, $1, x, \ldots, x^{n-1}$ is a $\mathbb{K}$-basis of $A$. We claim that the only ideals of $A$ are $Ax^i$, for $0 \leq i \leq n$. Indeed, an ideal $I$ of $A$ has the form $I = J/(X^n)$, where $J$ is an ideal of $\mathbb{K}[X]$ containing $X^n$. But $\mathbb{K}[X]$ is a PID, and so $J = (f)$, for $f \in \mathbb{K}[X]$ monic, and $f|X^n$. Hence, $f = X^i$ for $0 \leq i \leq n$, which proves the claim.

Define

$$
\lambda : A \rightarrow \mathbb{K},
$$

$$
x^i \mapsto 1.
$$

Clearly, $\lambda$ is symmetric, and $\ker \lambda$ doesn’t not contain any nonzero ideal of $A$. \hfill \Box

Remark 8.1.3. The algebra $A = \mathbb{K}[X]/(X^n)$ is called the algebra of truncated polynomials.

Theorem 8.1.4. $A = \mathbb{K}[X]/(X^n)$ has finite representation type. More precisely :

1. the only indecomposable $A$-modules (up to isomorphism) are $A/Ax^i$, for $1 \leq i \leq n$ ;

2. $A$ is uniserial, i.e. any module has a unique composition series.
Proof. By induction on $n$. If $n = 1$, then $A \cong \mathbb{K}$, and the result is obvious. Suppose now $n \geq 2$, and let $M$ be a finitely generated $A$-module.

Suppose that there exists $m_1 \in M$ such that $x^{n-1}m_1 \neq 0$. Then $m_1$ generates a submodule $Am_1$ with basis $m_1, xm_1, \ldots, x^{n-1}m_1$. Indeed, if $\sum_{i=0}^{n-1} \lambda_ix^im_1 = 0$, then $x^{n-1}\sum_{i=0}^{n-1} \lambda_ix^im_1 = \lambda_0x^{n-1}m_1 = 0$, and so $\lambda_0 = 0$, and repeat with $x^{n-2}$, etc. Therefore $Am_1$ is a free module isomorphic to $AA$, which is itself injective (by the previous lemma). Hence $Am_1$ is injective as well, and the injection $Am_1 \hookrightarrow M$ has a retraction, hence $M = Am_1 \oplus M_2$. Suppose that there exists $m_2 \in M_2$ such that $x^{n-1}m_2 \neq 0$. Then by the same argument, we have $M = Am_1 \oplus Am_2 \oplus M_3$. Continuing in this way until the initial assumption is false leads to a decomposition

$$M \cong Am_1 \oplus \cdots \oplus Am_k \oplus N,$$

where $N$ is a submodule such that $x^{n-1}N = 0$.

Hence $N$ can be viewed as a $B$-module, where $B = A/Ax^{n-1} \cong \mathbb{K}[X]/(X^{n-1})$. By induction, the only indecomposable $B$-modules are $B/Bx^i$, for $1 \leq i \leq n-1$. Therefore $N \cong \bigoplus_i (B/Bx^i)^{\oplus k_i}$. Clearly, $B/Bx^i \cong (A/Ax^{n-1})/(Ax_i/Ax^{n-1}) \cong A/Ax^i$. We view $B/x^iB$ as a left $A$-module (with $x_{n-1}$ acting by 0).

This proves that $M$ decomposes as a direct sum of modules of the form $A/Ax^i$. We need to show that each module $A/Ax^i$ is indeed indecomposable. We know that $\text{end}_R R \cong R^{\text{op}}$, for any ring $R$. In particular, we have an isomorphism

$$\text{end}_A(A/Ax_i) \cong (A/Ax_i)^{\text{op}} = A/Ax_i$$

$$f \mapsto f(\overline{1})$$

$$m \mapsto \overline{a}.$$

Now $A/Ax_i \cong \mathbb{K}[X]/(X^i)$ is a local ring with maximal ideal $Ax/AX^i$ because the only maximal ideal of $\mathbb{K}[X]$ containing $X^i$ is $(X)$. Hence, since $\text{end}_A(A/Ax_i)$ is local, we have that $A/Ax_i$ is indecomposable. This proves that $A$ has finite representation type, with $n$ indecomposable modules (up to isomorphism).

For uniseriality\footnote{Is this even a real word?}, remark that the only ideals of $\mathbb{K}[X]$ containing $X^n$ are $(X^n) \supset \cdots \supset (X)$. Therefore, the only ideals of $A = \mathbb{K}[X]/(X^n)$ are $A \supset Ax \supset \cdots \supset Ax^{n-1} \supset 0$. This proves that $AA$ is uniserial, and so are any of its quotients $A/Ax_i$.

\[\square\]

Remark 8.1.5. $Ax^i$ is an indecomposable submodule of $AA$. It is in fact it is
isomorphic to $A/Ax^{n-i}$ with
\[
\begin{align*}
A/Ax^{n-i} & \rightarrow Ax^i \\
\bar{I} & \mapsto x^i \\
\bar{x} & \mapsto x^{i+1} \\
& \vdots \\
\bar{x}^{n-i} & \mapsto x^n = 0.
\end{align*}
\]

### 8.2 Group algebras of finite representation type

Let $G$ be a finite group. Then $A = \mathbb{K}G$ is finite dimensional.

1. If $\text{char} \, \mathbb{K} = 0$ or $\text{char} \, \mathbb{K} = p$ and $p \nmid |G|$, then by Maschke theorem, $\mathbb{K}G$ is semisimple. Then every indecomposable module is simple and in particular, there are finitely many of them, up to isomorphism. Thus $\mathbb{K}G$ is of finite representation type.

2. If $\text{char} \, \mathbb{K} = p$, and if $G$ is a $p$-group, i.e. $|G| = p^n$ for some $n \in \mathbb{N}^*$, then we obtain theorem 8.2.2.

3. If $\text{char} \, \mathbb{K} = p$ which divides $|G|$, then some standard method called induction and restriction allow to pass from a $p$-Sylow subgroup of $G$ to the whole of $G$, and we obtain theorem 8.2.3.

**Lemma 8.2.1.** Let $G$ be a finite $p$-group, with $p$ prime.

1. Any maximal subgroup of $G$ is normal and has index $p$.

2. If there is a unique maximal subgroup, then $G$ is cyclic.

3. If there is at least two distinct maximal subgroups, then $G$ has a quotient isomorphic to $C_p^2$.

**Proof.** 1. By induction on $n$, where $|G| = p^n$. If $n = 1$, then $G$ is cyclic and 1 is the only maximal subgroup, which of course has index $p$.

Suppose $n \geq 2$, and let $M$ be a maximal subgroup. We use the fact that $Z = Z(G)$ is non trivial (consequence of the class equation). We have two cases:

(a) $Z \leq M$, then $M/Z$ is a maximal subgroup of $G/Z$. Since $|G/Z| < |G|$, induction tells us that $M/Z$ is normal, and has index $p$. It follows that $M$ is normal in $G$ with index $p$.

(b) $Z \nsubseteq M$, then $M < ZM$, and the later is a subgroup since $Z$ is normal. So $ZM = G$ by maximality of $M$. Let $g \in G$, then $g = zm$ with $z \in Z$, $m \in M$. Then $gMg^{-1} = zmMm^{-1}z^{-1} = \ldots$
$zMz^{-1} = M$. Hence $M$ is normal. Moreover $GM = H$ is a group without any subgroup apart from 1 and $H$. Hence $H \cong C_p$ and $M$ has index $p$.

2. Suppose $M$ is the unique maximal subgroup of $G$. Let $g \in G \setminus M$. Then $(g)$ is not contained in $M$, hence not contained in any maximal subgroup, hence $(g) = G$, and $G$ is cyclic.

3. Suppose that $M_1$ and $M_2$ are two distinct maximal subgroups of $G$. Then $M_1 < M_1 M_2 \leq G$, and by maximality of $M_1$ we have $M_1 M_2 = G$. Let $N = M_1 \cap M_2$. By the second isomorphism theorem, we have $G/M_1 \cong M_2/N$ and similarly, $G/M_2 \cong M_1/N$. The obvious group homomorphism $G \rightarrow G/M_1 \times G/M_2 \cong C^2_p$ has kernel $N$. Therefore it induces an injective map $G/N \rightarrow C^2_p$. However $|G/N| = p^2$ and so $G/N \cong C^2_p$.

\[ \square \]

**Theorem 8.2.2.** With the assumptions of point 2., i.e. char $K = p$ and $G$ is a $p$-group, $KG$ has finite representation type if and only if $G$ is cyclic. More precisely,

1. If $G$ is cyclic, then $KG$ is uniserial and there are $|G|$ indecomposable modules up to isomorphism.

2. If $G$ is not cyclic, then $KG$ has a quotient isomorphic to $K[X, Y]/(X^2, Y^2, XY)$, which has infinite representation type.

**Proof.** 1. If $G$ is cyclic, let $x = g^{-1} \in KG$, where $g$ is a generator of $G$, and define

$$
\phi : K[X] \rightarrow KG \\
X \mapsto x = g^{-1}.
$$

It is a surjective algebra homomorphism because $\phi(X + 1) = g$, $\phi((X + 1)^k) = g^k$, and im $\phi$ contains a basis of $KG$. Now $x^{p^n} = (g^{-1})^{p^n} = g^{p^n} - 1 = 0$, so ker $\phi \geq (X^{p^n})$, and $\phi$ induces a surjective algebra homomorphism $A = K[X]/(X^{p^n}) \rightarrow KG$. We have dim$_K A = \dim_K KG = p^n$, so $A \cong KG$. We know by theorem 8.1.4 that $A$ is uniserial and of finite representation type.

2. Suppose $G$ not cyclic. By lemma 8.2.1, there is a normal subgroup $N < G$ such that $G/N \cong C^2_p$. Therefore there is a surjective algebra homomorphism $\phi : KG \rightarrow K(C^2_p)$, and we have an isomorphism (same argument as before)

$$
K[X, Y]/(X^p, Y^p) \rightarrow K(C^2_p) \\
X \mapsto g - 1 \\
Y \mapsto h - 1,
$$
where \( g \) is a generator of \( C_p \times 1 \) and \( h \) is a generator of \( 1 \times C_p \). Now has a quotient \( B = \mathbb{K}[X,Y]/(X^2,Y^2,XY) \). So \( \mathbb{K}G \) has a quotient isomorphic to \( B \). It is shown in example 4.2.6 that \( B \) has infinite representation type, provided that \( \mathbb{K} \) is infinite. If \( \mathbb{K} \) is finite, then the same result hold (see exercise 6 of sheet 13). It follows that \( \mathbb{K}G \) also have infinite representation type (because any indecomposable module of a quotient \( \mathbb{K}G/I \) remains indecomposable seen as a module over the base ring, on which \( I \) acts as 0).

\[
\text{Theorem 8.2.3. With the assumptions of point 3., i.e. } \text{char } \mathbb{K} = p \mid |G|, \mathbb{K}G \text{ has finite representation type if and only if the } p\text{-Sylow subgroup of } G \text{ are cyclic.}
\]

\textit{Proof.} Not treated in this course.

8.3 Quivers of finite representation type

Let \( Q \) be a finite quiver. Suppose that \( Q \) has no oriented cycles, so that \( \mathbb{K}Q \) is finite dimensional. Associated with \( Q \) there is an unoriented graph \( \bar{Q} \) which is \( Q \) with arrows replaced by unoriented edges.

\textbf{Theorem 8.3.1 (Gabriel, 1972).} \( \mathbb{K}Q \) has finite representation type if and only if the undirected graph \( \bar{Q} \) is a disjoint union of Dynkin graph.

\textit{Proof.} Not treated in this course.

This concludes this course about finite dimentional algebra, and is also the last course of prof. J. Thévenaz given to math students!
Bibliography